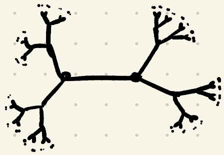
2024-11-28 LECTURE 27 MA3F1 SAUL SCHLEIMER

① TREES

DEF. A GRAPH IS A ONE-DIM'L CW-COMPLEX.

EXAMPLES:  $\cong S^1$  R_2  T_3 } REGULAR THREE-VALENT TREE

LEMMA: SUPPOSE T IS A CONNECTED NONEMPTY GRAPH: THE FOLLOWING ARE EQUIVALENT:

(i) $\pi_1(T) \cong 1\!\!1$

(ii) T IS CONTRACTIBLE (ii') T DEF RETRACTS TO A POINT

(iii) THERE IS NO EMBEDDED LOOP IN T

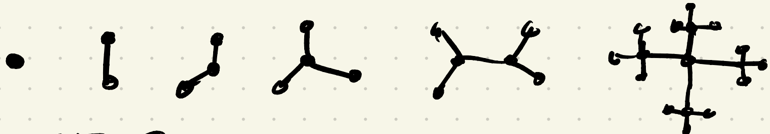
(iv) FOR ANY $x, y \in T$ THERE IS A UNIQUE EMBEDDED PATH FROM x TO y IN T.

(v) REMOVING ANY (OPEN) EDGE FROM T DISCONNECTS T.

FOR FINITE T (vi) $\# \mathrm{VERT}(T) - \# \mathrm{EDGE}(T) = 1$.] $\chi(T) = 1$

(vii) $T = \{pt\}$ OR a) T HAS A LEAF AND b) T - ANY LEAF IS A TREE

DEF: SUPPOSE T IS SUCH A GRAPH: WE CALL T A TREE.

EXAMPLES:  ETC.

② SPANNING TREES

DEF: SUPPOSE X IS A CONN. GRAPH. SUPPOSE $T \subset X$ HAS

(i) $X^{(0)} \subset T$ AND

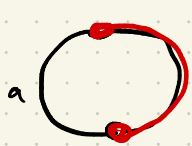
(ii) T IS A TREE

} THEN CALL T A SPANNING TREE FOR X.

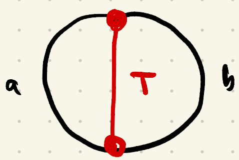
LEMMA: ALL CONNECTED GRAPHS CONTAIN SPANNING TREES.
[THIS IS EQUIVALENT TO AX of CHOICE!]

PROPOSITION: SUPPOSE X IS A CONNECTED GRAPH. SUPPOSE $T \subset X$ IS A SPANNING TREE. LET $S = X - T$ BE THE SET of OPEN "NON-TREE" EDGES.
THEN $\pi_1(X) \cong F_S$. THE FREE GROUP GENERATED BY S.

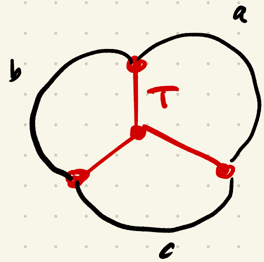
EXAMPLES



$\pi_1 \cong \mathbb{Z}$



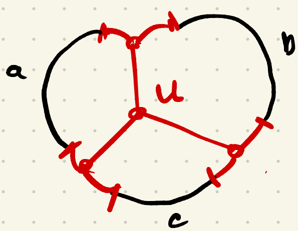
$\pi_1 \cong F_2$
$\cong \langle a, b \mid \rangle$

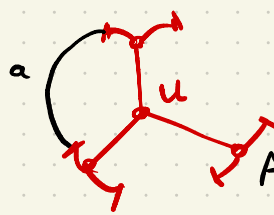


$\pi_1 \cong F_3$.
$\cong \langle a, b, c \mid \rangle$.

PROOF of PROP: LET $U \subset X$ BE A NEIGHBOURHOOD of T IN X WHICH DEF RETRACTS TO T. [PROP A3]
NOTE $\pi_1(U) \cong \pi_1(T) \cong \mathbb{1}$ BECAUSE T IS A TREE.
FOR EVERY $s \in S$ [NON-TREE EDGE] DEFINE
$A_s = U \cup s$.

PICTURE





A_a

NOTE $A_r \cap A_s$ AND $A_r \cap A_s \cap A_t$ ARE PATH CONNECTED, AND $\pi_1(A_s) \cong \mathbb{Z}$. APPLY SvK TO FIND

$$\pi_1(X) \cong \mathop{\ast}_{s \in S} \mathbb{Z} = F_S, \quad \text{AS DESIRED.} \qquad \square$$

② FROM PRESENTATIONS TO COMPLEXES

SUPPOSE $G \cong \langle S \mid R \rangle$ IS A PRESENTATION of a GROUP. WE BUILD X_G, THE PRESENTATION TWO-COMPLEX AS FOLLOWS:

0) $X^{(0)} = \{x_0\}$ IS A POINT.

1) $X^{(1)}$ IS THE ROSE R_S [ONE PETAL e^1_s FOR EACH $s \in S$]

2) THERE IS A TWO CELL D^2_r FOR EACH $r \in R$ WITH $\varphi_r : S^1_r \to X^{(1)}$ THE CONCAT. of THE EDGES SPELLING $r \in R$.

PROP: $\pi_1(X_G) \cong G$.

COROLLARY: EVERY GROUP G ARISES AS A FUND. GROUP.

PROOF: SET $S = G$ AND $R = \{ a * b * (ab)^{-1} \mid a, b \in G \}$. SO $G \cong \langle S \mid R \rangle$ ∎

EXAMPLE:

$\pi_1(X) \cong \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$.