2024-12-03 SAULSCHLEIMER MA3F1 LECTURE 29

(1) QUESTION ASKED YESTERDAY

DEF: AN n-MANIFOLD M^n IS A TOP. SPACE WHICH IS

(i) HAUSDORFF,

(ii) SECOND COUNTABLE, AND

(iii) LOCALLY $\mathbb{R}^n$.

EXAMPLE: S^n, T^n, $\mathbb{RP}^n$, SUBMANIFOLDS, PRODUCTS, CONNECT SUMS.

QUESTION: DO ALL MANIFOLDS HAVE CW-COMPLEX STRUCTURES?

ANSWER: YES IN DIMENSION $n \neq 4$

OPEN IN DIMENSION $n=4$.

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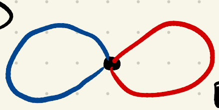
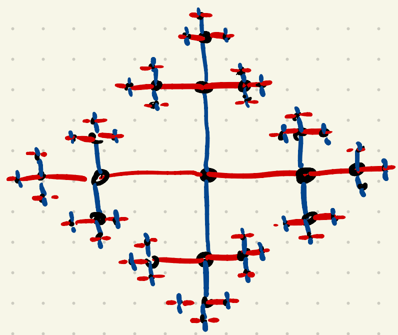
(2) UNIVERSAL COVERS

SUPPOSE (X, x_0), $(\tilde{X}, \tilde{x}_0)$ ARE POINTED AND PATH-CONN.

SUPPOSE $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ IS A COVERING MAP. WE SAY p IS A UNIVERSAL COVERING MAP, AND $(\tilde{X}, \tilde{x}_0)$ IS A UNIVERSAL COVER of (X, x_0) IF $\pi_1(\tilde{X}, \tilde{x}_0) \cong \mathbb{1}$.

EXAMPLES ① $p: \mathbb{R}^n \to T^n$

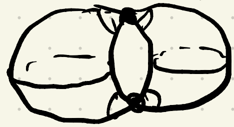
② $p: S^n \to \mathbb{RP}^n$

③  R_2 HAS UNIVERSAL COVER 

NON-EXAMPLE THE BARRING SPACE  HAS NO UNIV COVER.

EXERCISES: BUILD UNIV. COVER of

① $\mathbb{RP}^2 \vee \mathbb{RP}^2$. ②



TWO POINT UNION of S^2 AND S^2.

THEOREM [PAGES 63-65 of HATCHER] SUPPOSE (X, x_0) IS A CONN. CW-COMPLEX THEN THERE EXISTS A UNIV. COVER $(\tilde{X}, \tilde{x}_0)$ AND $(\tilde{X}, \tilde{x}_0)$ IS UNIQUE (UP TO ISOMORPHISM).

PLAN of PROOF: SUPPOSE (X, x_0) CW, PATH-CONNECTED.

① BUILD SET $\tilde{X}$, POINT $\tilde{x}_0$

② DEFINE FUNCTION $p : \tilde{X} \to X$. CHECK $p(\tilde{x}_0) = x_0$.

③ DEFINE TOPOLOGY ON $\tilde{X}$.

④ CHECK p IS A COVERING MAP.

⑤ PROVE $\tilde{X}$ IS PATH CONNECTED.

⑥ PROVE $\pi_1(\tilde{X}, \tilde{x}_0)$ IS TRIVIAL.

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③ BUILD $\tilde{X}$.

AGAIN (X, x_0) IS CW, PATH-CONN.

DEFINE

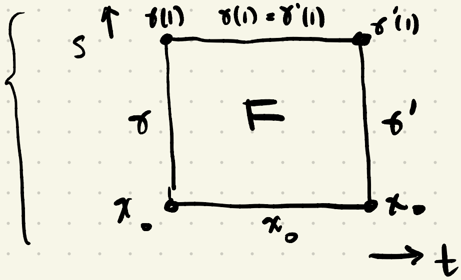
$$\mathrm{PATHS}(X, x_0) = \left\{ \gamma : I \to X \;\middle|\; \begin{array}{l} \gamma \text{ CONTINUOUS} \\ \gamma(0) = x_0 \end{array} \right\}$$

NOTE $\mathrm{LOOPS}(X, x_0) \subset \mathrm{PATHS}(X, x_0)$ BUT IF $X \neq \{pt\}$ THEN INCLUSION IS STRICT.

RECALL $\gamma \underset{\partial}{\simeq} \gamma'$ IF THERE IS A HOMOTOPY IN X

$F : I \times I \longrightarrow X$

WITH



<u>DEFINE</u> $\widetilde{X} = \{ [\gamma] \mid \gamma \in \text{PATHS}(X, x_0) \}$.

<u>DEFINE</u> $\widetilde{x}_0 = [e]$ THE CLASS of THE CONSTANT PATH.

<u>NOTE</u>: $\pi_1(X, x_0) \subset \widetilde{X}$.

(4) WE DEFINE $p : \widetilde{X} \longrightarrow X$ BY $p([\gamma]) = \gamma(1)$.

THIS IS WELL-DEFINED BY DEF of $\underset{\partial}{\simeq}$.

NOTE $p([e]) = e(1) = x_0$. SO $p(\widetilde{x}_0) = x_0$.

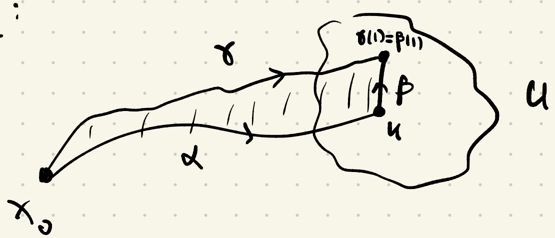
(5) <u>TOPOLOGY ON $\widetilde{X}$.</u>

SUPPOSE $U \subset X$ IS CONTRACTIBLE, OPEN. SUPPOSE $u \in U$. FIX $\alpha : I \longrightarrow X$ WITH $\alpha(0) = x_0$, $\alpha(1) = u$.

DEFINE $U_\alpha \subset \widetilde{X}$ TO BE

$$U_\alpha = \left\{ [\gamma] \in \widetilde{X} \;\middle|\; \begin{array}{l} \text{THERE IS } \beta \in \text{PATHS}(U, u) \\ \text{SO THAT } \gamma \underset{\partial}{\simeq} \alpha * \beta \end{array} \right\}$$

<u>PICTURE</u>:



<u>LEMMA</u>: $\{ U_\alpha \}$ IS A BASIS FOR A TOPOLOGY ON $\widetilde{X}$.

THAT IS: GIVEN U_α, V_β, THEIR INTERSECTION

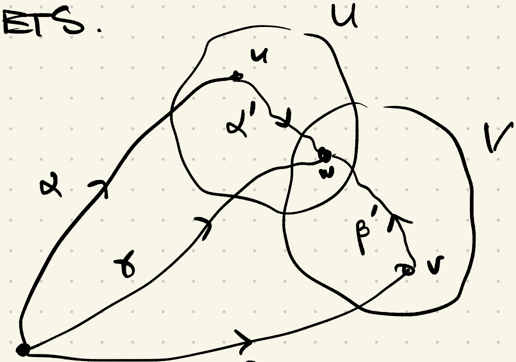
IS A UNION of SUCH SETS.

<u>PROOF</u>: FIX $\gamma \in U_\alpha, V_\beta$

SO THERE ARE

$\alpha' \in PATHS(U, u)$

$\beta' \in PATHS(V, v)$



SO THAT $\alpha * \alpha' \overset{\partial}{\simeq} \gamma \overset{\partial}{\simeq} \beta * \beta'$ x_0

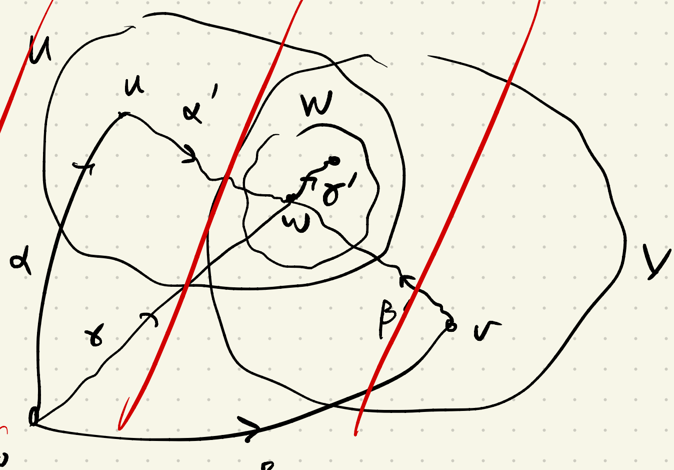
LET $w = \gamma(1)$. PICK $W \subset U \cap V$ CONTRACTIBLE SO THAT $w \in W$. [RECALL X CW SO LOCALLY CONTRACTIBLE] NOTE $\gamma \in W_\gamma$.

SUPPOSE $\gamma' \in PATHS(W, w)$. SO $[\gamma * \gamma'] \in W_\gamma$.

NOTE $\gamma \overset{\partial}{\simeq} \alpha * \alpha'$. THUS $\gamma * \gamma' \underset{\partial}{\simeq} \alpha * \alpha' * \gamma'$

AND $\alpha' * \gamma' \in PATHS(U, u)$. SO $[\gamma * \gamma'] \in U_\alpha$.

<u>PICTURE</u>



SIMILARLY, $[\gamma * \gamma'] \in V_\beta$. SO $W_\gamma \subset U_\alpha \cap V_\beta$. $\square$

(6) <u>$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ IS A COVERING MAP.</u>

p IS CONTINUOUS: SUFFICES TO CHECK

$$p^{-1}(U) = \bigcup_{\gamma(1)\in U} U_\gamma$$

IS OPEN.

p IS A COVERING: WE MUST SHOW $p^{-1}(U)$ IS A DISJOINT UNION of COPIES of U.