THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: APRIL 2025

INTRODUCTION TO TOPOLOGY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER ALL 4 QUESTIONS.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. Suppose that (X, x_0) is a pointed topological space.

a) Define a *based loop* in (X, x_0). **[2]**

Let $\mathrm{Loops}(X, x_0)$ be the set of based loops in (X, x_0). Suppose that α, β, and γ are three such.

b) Define the *constant loop* e in (X, x_0). **[2]**

c) Define what it means for there to be a *homotopy relative to the basepoint* between α and β. We denote this relation by $\alpha \simeq \beta$. **[2]**

d) Sketch a proof that homotopy relative to the basepoint is an equivalence relation on $\mathrm{Loops}(X, x_0)$. (You may draw pictures or give parametrisations as you prefer.) **[4]**

e) Define the *concatenation* $\alpha * \beta$ of α and β. **[2]**

f) Define the *reverse* $\bar{\alpha}$ of α. **[2]**

g) Suppose that $e \in \mathrm{Loops}(X, x_0)$ is the constant loop. Sketch proofs for each of the following. (You may draw pictures or give parametrisations as you prefer.)

(i) $e * \alpha \simeq \alpha * e \simeq \alpha$ **[3]**

(ii) $\alpha * \bar{\alpha} \simeq \bar{\alpha} * \alpha \simeq e$ **[4]**

(iii) $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ **[4]**

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2. Suppose that X and Y are topological spaces. Suppose that $\gamma\colon Y \to X$ is a continuous function.

a) Define what it means for γ to be a *covering map*. **[3]**

b) Prove that $s\colon S^1 \to S^1$, given by $s(z) = z^2$, is a covering map. Clearly state any results from lecture which you use. **[6]**

c) Sketch a proof that $p\colon \mathbb{R} \to S^1$, given by $p(t) = \exp(2\pi i t)$, is a covering map. **[3]**

d) Suppose that $\gamma\colon Y \to X$ is a covering map. Define the *deck group* of γ. **[3]**

e) With $s\colon S^1 \to S^1$ as in part (b), determine the deck group of s. Prove that your answer is correct, clearly stating any results from lecture which you use. **[7]**

f) With $p\colon \mathbb{R} \to S^1$ as in part (c), determine the deck group of p. Sketch a proof that your answer is correct. **[3]**

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3. Suppose that X is a topological space and $A \subset X$ is a subspace. Suppose that x_0 is a point of A.

a) Define what it means for a map $r \colon X \to A$ to be a *retraction*. **[2]**

b) Suppose that $r \colon X \to A$ is a retraction. Prove that the induced homomorphism $r_* \colon \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective. **[3]**

c) Give the definition of $\vee_\infty S^1$, the one-point union of a countable collection of pointed circles. **[2]**

d) Give the definition of $E \subset \mathbb{R}^2$, the earring space. **[2]**

e) Determine if $\pi_1(\vee_\infty S^1)$ is countable or uncountable. Prove your answer is correct, clearly stating any results from lecture which you use. **[7]**

f) Determine if $\pi_1(E)$ is countable or uncountable. Prove your answer is correct, clearly stating any results from lecture which you use. **[7]**

g) Using the above or otherwise, decide if $\vee_\infty S^1$ is homotopy equivalent to E. Sketch a proof that your answer is correct. **[2]**

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4. For this question we take Σ_2 to be the compact, connected, oriented surface of genus two (without boundary). (In the lectures this was defined as the connect sum of two copies of the two-torus.)

a) Draw a picture of Σ_2. **[2]**

b) Give a CW complex structure X for Σ_2. Also, indicate how X relates to the picture given in part (1). **[6]**

c) Draw $X^{(1)}$, the one-skeleton of X. Choose a spanning tree T for $X^{(1)}$. Label T in your picture of $X^{(1)}$. Label and orient all of the non-tree edges. **[4]**

d) Give a presentation for $\pi_1(X^{(1)})$. Prove your answer is correct, clearly stating any results from lecture which you use. **[4]**

e) Draw the two-cells of $X^{(2)}$, the two-skeleton of X. Label the boundary edges either as tree or non-tree edges; label and orient the latter according to the conventions introduced above. **[4]**

f) Give a presentation for $\pi_1(\Sigma_2)$. Prove your answer is correct, clearly stating any results from lecture which you use. **[5]**
