Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 4.1. Let Q_n be the space of n points, equipped with the discrete topology. Find a two-fold cover $p\colon Q_4 \to Q_2$ and compute its deck group.

Exercise 4.2. Define $\mathbb{RP}^2$ to be the space of lines through the origin in $\mathbb{R}^3$. That is, $\mathbb{RP}^2$ is the quotient space obtained from $\mathbb{R}^3 - \{0\}$ by identifying two vectors u, v if and only if there is a non-zero real λ so that $u = \lambda v$. Construct a two-fold covering map $p\colon S^2 \to \mathbb{RP}^2$.

Exercise 4.3. Let F be the *figure-eight graph*: the graph with one vertex and two edges. List all connected two- and three-fold covers of F, up to isomorphism. Sketch an argument showing that your lists are complete and irredundant.

Exercise 4.4. For each of the covers of F given in Exercise 4.3 compute the resulting deck group.

Exercise 4.5. With notation as set in class: check the following claims, needed in the proof that Φ is a homomorphism.

- Show that $\widetilde{\omega}_{m+n} \overset{\partial}{\simeq} \widetilde{\omega}_m * (\tau_m \circ \widetilde{\omega}_n)$.
- Suppose that $\alpha \overset{\partial}{\simeq} \beta$ are paths in $\mathbb{R}$. Show that $p \circ \alpha \overset{\partial}{\simeq} p \circ \beta$ as paths in S^1.
- Suppose that α and β are paths in $\mathbb{R}$ with $\alpha(1) = \beta(0)$. Show that $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$.

Exercise 4.6. [Hard.] Let $T^2 = S^1 \times S^1$ be the two-torus. For each $d > 0$, count the isomorphism classes of connected d-fold covers of T^2.

Exercise 4.7. [Problem 6, page 38, Hatcher.] We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, 1) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi\colon \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X, x_0)$, when X is path-connected.

Exercise 4.8. Suppose that $p\colon \widetilde{X} \to X$ is a covering map. Suppose that $x_0 \in X$ and $e\colon I \to X$ is the constant path at x_0. Give a direct proof (without using Proposition 1.30) that any lift $\widetilde{e}$ of e is again a constant path.

Exercise 4.9. [Medium] Boil the proof of Proposition 1.30 all the way down to prove that covering maps have the *path lifting property*: suppose that $p\colon \widetilde{X} \to X$ is a covering map, suppose that $f\colon I \to X$ is a path, and set $x_0 = f(0)$. Suppose that $\widetilde{x}_0 \in \widetilde{X}$ is sent to x_0 by p. Then there is a unique lift $\widetilde{f}$ of f so that $\widetilde{f}(0) = \widetilde{x}_0$.