Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 10.1. Suppose that p and q are points of $\mathbb{D}$. Find all linear fractional transformations in $\mathrm{AUT}(\mathbb{D})$ sending p to q. ◇

Exercise 10.2. Suppose that $f\colon \mathbb{H} \to \mathbb{H}$ is holomorphic and that $f(i) = i$. Prove that $|f'(i)| \leq 1$. Prove, moreover, that if $|f'(i)| = 1$ then there is some angle θ so that:

$$f(z) = \frac{\cos(\theta)z - \sin(\theta)}{\sin(\theta)z + \cos(\theta)}$$ ◇

Exercise 10.3. For each domain U below do the following.

i) Give a sketch. Draw and label the real and imaginary axes, orient and label the arcs of ∂U, lightly shade the interior of U, and label points of interest.

ii) Give a biholomorphism from U to the unit disc $\mathbb{D}$, expressed as a composition of simple functions. Draw a copy of $\mathbb{D}$; label the images of the boundary arcs of U and the images of the points of interest.

Here are the domains.

a) The quarter plane $Q = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z),\ 0 < \mathrm{REAL}(z)\}$.

b) The upper half disc $P = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z),\ |z| < 1\}$.

c) The lune $L = \{z \in \mathbb{C} : |z - i| \leq \sqrt{2},\ |z + i| \leq \sqrt{2}\}$.

d) The quarter disc $A = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z),\ 0 < \mathrm{REAL}(z),\ |z| < 1\}$.

e) The slit disc $D = \mathbb{D} - [0, 1)$.

f) The bi-infinite strip $B = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z) < 1\}$.

g) The half-strip $H = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z) < 1,\ 0 < \mathrm{REAL}(z)\}$. ◇

Exercise 10.4. Let $S = \{z \in \mathbb{C} : 0 < \mathrm{IMAG}(z) < 1\}$ be the bi-infinite strip. Find all automorphisms (biholomorphic maps) from the strip to itself that fix the point at infinity. ◇

Exercise 10.5. [Not examinable.] Let $A = A(0; 1/2, 2)$ be the annulus centred on the origin with inner radius $1/2$ and outer radius 2. Find all automorphisms (biholomorphic maps) from A to itself. ◇