Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 7.1. Suppose that U is a domain and that $z_0 \in U$. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic. Suppose that $f(z_0) \neq 0$ and that g has a simple zero at z_0. Take $h = f/g$.

- Prove that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic.
- Prove that h has a simple pole at z_0.
- Prove that $\textsc{Res}(h, z_0) = f(z_0)/g'(z_0)$. ◇

Exercise 7.2. Suppose that we are in the same setting as Exercise 7.1, except now g has a zero of order two at z_0. Prove that

$$\textsc{Res}(h, z_0) = \frac{2}{3(g''(z_0))^2}\Big(3f'(z_0)g''(z_0) - f(z_0)g'''(z_0)\Big)$$

[You should check this in the special case where $f(z) = a + bz$, where $g(z) = cz^2 + dz^3$, and where $a, b, c, d \in \mathbb{C}$ with $c \neq 0$.] ◇

Exercise 7.3. Suppose that $p, q \in \mathbb{R}[x]$ are polynomials so that $\textsc{Deg}(q) \geq \textsc{Deg}(p) + 2$ and so that q has no roots in $\mathbb{R}$. Take $h = p/q$ to be the resulting rational function.

- Prove that $I = \int_{-\infty}^{\infty} h\, dx$ converges.
- Prove that p and q have unique holomorphic extensions, P and Q, to all of $\mathbb{C}$. Prove that P and Q are polynomials.
- Take $H = P/Q$. Prove that $I = 2\pi i \cdot \sum \textsc{Res}(H, w)$; here the sum is over the zeros of Q in the upper half plane. ◇

For each integral below do an appropriate combination of the following: check convergence; consider the possibility of simplifications and/or symmetries; choose an appropriate contour and meromorphic function; sketch the contour, labelling the arcs and points of interest; compute residues; estimate away "minor" arcs; and check your answer using a numerical integrator.

Exercise 7.4. Compute

$$I_2 = \int_{-\infty}^{\infty} \frac{dz}{(1+x^2)^2}$$ ◇

Exercise 7.5. Suppose that $a > 0$ is real. Compute the following improper integrals:

$$I_C(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, dx \qquad \text{and} \qquad I_S(a) = \int_{-\infty}^{\infty} \frac{\sin(ax)}{1+x^2}\, dx$$ ◇

Exercise 7.6. [Not examinable] Compute the following improper integrals, for $n \geq 1$:

$$I_n = \int_{-\infty}^{\infty} \frac{dz}{(1+x^2)^n}$$ ◇