Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 8.1. Define $R\colon [0,\infty) \to \mathbb{C}$ by $R(t) = t\exp(2\pi i t)$. Define $U = \mathbb{C} - \operatorname{Im}(R)$.

- Sketch the image of R in $\mathbb{C}$. Additionally, draw and label the real and imaginary axes; label points of intersection between the axes and the image of R.
- Prove that U is a domain. Prove that U is homeomorphic to $\mathbb{C} - \mathbb{R}_{\geq 0}$. Deduce that $H_1(U) \cong 0$.
- Let g be the branch of the logarithm on U so that $g(1/2) = \ln(1/2)$. Find $g((2k+1)/2)$ for all $k \in \mathbb{N}$. ◇

For each integral below do an appropriate combination of the following: check convergence; consider the possibility of simplifications and/or symmetries; choose an appropriate contour and meromorphic function; sketch the contour, labelling the arcs and points of interest; compute residues; estimate away "minor" arcs; and check your answer using a numerical integrator.

Exercise 8.2. For all $k \in \mathbb{N}$ define:

$$I_k = \int_0^\infty \frac{\ln^k(x)}{1+x^2}\,dx$$

Prove that I_k vanishes when k is odd. Compute I_4. ◇

Exercise 8.3. For all real a with $-1 < a < 1$ define:

$$J(a) = \int_0^\infty \frac{x^a}{1+x^2}\,dx$$

Prove that $J(-a) = J(a)$. Compute $J(a)$. [Hint: $x^a = \exp(a \cdot \ln(x))$.] ◇

Exercise 8.4. Suppose that I_k is as defined in Exercise 8.2. Suppose that $J(a)$ is as defined in Exercise 8.3. Prove that

$$J(a) = \sum_{k=0}^\infty \frac{I_k}{k!} a^k$$

[Hint: $x^a = \exp(a \cdot \ln(x))$.] ◇

Exercise 8.5. [Harder.] Compute

$$S = \int_{-\infty}^\infty \frac{\sin(x)}{x}\,dx$$ ◇