Please let me (Saul) know if any of the problems are unclear or have typos.

Exercise 9.1. Prove that $\mathbb{C}^\times$ and $\mathbb{D}^\times$ are homeomorphic. Prove that they are not conformally equivalent. ◇

Exercise 9.2. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and injective. Set $V = f(U)$. Prove that V is a domain. Prove that f is a conformal equivalence between U and V. [The proof of this is in the lecture notes, spread across several sections.] ◇

Exercise 9.3. Prove that $\hat{\mathbb{C}}$, the extended plane, is homeomorphic to S^2, the unit two-sphere in $\mathbb{R}^3$. ◇

Exercise 9.4. Suppose that a, b, c, and d are complex numbers with $ad - bc \neq 0$. Let $f(z) = (az+b)/(cz+d)$ be the resulting linear fractional transformation. Express f as a composition of additions A_w, multiplications M_w, and inversions V. [Careful: the proof splits into cases.] ◇

Exercise 9.5. Suppose that $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic. Prove that f is *rational*: a ratio of polynomials. ◇

Exercise 9.6. For each of the following choices of $T \subset \hat{\mathbb{C}}$ find all automorphisms of $\hat{\mathbb{C}}$ preserving T setwise. Also, determine the isomorphism type of the resulting group $\mathrm{Aut}_T(\hat{\mathbb{C}})$.

a) $T = \{\infty, 0, 1\}$.

b) Fix $n > 2$, set $\omega = \mathrm{Exp}(2\pi i/n)$, and take $T = T(n) = \{\omega^k \mid k \in \mathbb{Z}\}$.

c) Set $\omega = \mathrm{Exp}(2\pi i/3)$ and take $T = \{1, \omega, \omega^2, \infty\}$. ◇

Exercise 9.7.

- Prove that $\mathrm{U}(1,1)$ is a group under matrix multiplication.
- Prove that every element M in $\mathrm{SU}(1,1)$ has the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

  for some $a, b \in \mathbb{C}$.
- Prove directly that $\mathrm{PSU}(1,1)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. ◇