2025-11-18 LECTURE 18 SAUL SCHLEIMER MA3B8

(1) TRICHOTOMY: IF $f: U^\times \to \mathbb{C}$ HOLOMORPHIC THEN THE MISSING POINT $z_0$ IS
(1) REMOVABLE SINGULARITY
(2) POLE of ORDER $0 < N < \infty$
(3) ESSENTIAL SINGULARITY.

(2) CASORATI-WEIERSTRASS THM

OR "ESSENTIAL SINGULARITIES ARE VERY SCARY" THM

THM: SUPPOSE $f: U^\times \to \mathbb{C}$ HOLOMORPHIC AND HAS AN ESSENTIAL SINGULARITY AT $z_0$. SUPPOSE $\varepsilon > 0$. SET $B^\times = B(z_0; \varepsilon) - \{z_0\}$. THEN $f(B^\times)$ IS DENSE IN $\mathbb{C}$.

PROOF: WE PROVE A CONTRAPOSITIVE. SUPPOSE $f(B^\times)$ NOT DENSE. SO HAVE $w \in \mathbb{C}$ AND $\delta > 0$ SO THAT $f(B^\times) \cap B(w; \delta) = \emptyset$. DEFINE $g: B^\times \to \mathbb{C}$ BY

$$g(z) = \frac{1}{f(z) - w}. \quad \text{SO } |g(z)| = \frac{1}{|f(z) - w|} \le \frac{1}{\delta} \text{ IS BOUNDED.}$$

SO $g$ HAS REMOVABLE SINGULARITY AT $z_0$. LET $G: B \to \mathbb{C}$ BE THE HOLOMORPHIC EXTENSION. SET $N = \mathrm{ORD}(G, z_0)$. NOTE $N \geq 0$. FACTOR TO FIND $G(z) = (z - z_0)^N H(z)$ FOR $H: B \to \mathbb{C}$ HOLOMORPHIC AND $H(z_0) \neq 0$. SO, ON $B^\times$

$$(z - z_0)^N H(z) = \frac{1}{f(z) - w} \quad \text{THUS } f(z) = w + \frac{1}{(z - z_0)^N H(z)}$$

NOTE $K = \frac{1}{H}$ IS HOLOMORPHIC ON SOME (SMALLER) $B' \subset B$.

SO $f(z) = w + \frac{1}{(z - z_0)^N} K(z)$. SO $\mathrm{ORD}(f, z_0) = -N$ AND $f$ HAS A POLE of FINITE ORDER (POSS. ZERO). $\square$

(3) MEROMORPHIC FUNCTIONS: WE DEFINE $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ TO BE THE EXTENDED PLANE. WE WILL GIVE $\hat{\mathbb{C}}$ ADDITIONAL STRUCTURE LATER. FIRST A COMPLICATED DEFINITION! [SORRY!]

DEF: SUPPOSE $U$ IS A DOMAIN. SUPPOSE $h : U \to \hat{\mathbb{C}}$ IS A FUNCTION. SUPPOSE $P(h) = h^{-1}(\infty)$ IS ISOLATED IN $U$. SUPPOSE $h|U - P(h)$ IS HOLOMORPHIC. SUPPOSE FOR EVERY $p \in P(h)$ WE HAVE $\varepsilon > 0$ AND $k > 0$ SO THAT SO $B = B(p;\varepsilon) \subset U$ AND $h|B^{\times}$ HAS A POLE OF ORDER $k$ AT $p$.

THEN WE CALL $h : U \to \hat{\mathbb{C}}$ MEROMORPHIC IN $U$.

WE CALL $P(h)$ THE SET OF POLES OF $h$.

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EXAMPLE: $f(z) = 1/z$ IS MEROMORPHIC IN $\mathbb{C}$.

LEMMA: SUPPOSE $f, g : U \to \mathbb{C}$ ARE HOLOMORPHIC. SUPPOSE NEITHER $f, g$ VANISHES IDENTICALLY IN $U$. THEN $h = f/g$ IS A MEROMORPHIC FUNCTION IN $U$ WITH

$$Z(h) = \{ z_0 \in U \mid \mathrm{ORD}(f, z_0) > \mathrm{ORD}(g, z_0) \}$$
$$P(h) = \{ z_0 \in U \mid \mathrm{ORD}(f, z_0) < \mathrm{ORD}(g, z_0) \}.$$

ALSO, FOR ALL $w \in U$ WE HAVE

$$\mathrm{ORD}(h, w) = \mathrm{ORD}(f, w) - \mathrm{ORD}(g, w).$$

PROOF: FIX $z_0 \in U$. SET $k = \mathrm{ORD}(f, z_0)$, $l = \mathrm{ORD}(g, z_0)$

FACTOR TO OBTAIN
$$f(z) = (z - z_0)^k F(z)$$
$$g(z) = (z - z_0)^l G(z)$$

WITH $F, G : U \to \mathbb{C}$ HOLOMORPHIC, $F(z_0) \neq 0$, $G(z_0) \neq 0$.

SO $H = F/G$ HOLOMORPHIC NEAR $z_0$.
WE NOW DEAL WITH CASES AS

* $l = 0$ OR
* $l > 0$ AND $k > l$, $k = l$, OR $k < l$.

IN ALL CASES FIND $h(z) = (z - z_0)^{k-l} H(z)$ NEAR $z_0$ ∎.

**EXAMPLE**: $p, q \in \mathbb{C}[z]$ POLYNOMIALS. THEN $h = p/q$ IS A RATIONAL FUNCTION AND IS MEROMORPHIC IN $\mathbb{C}$.

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WEIERSTRASS PRODUCT THM: IN FACT (FOR DOMAINS $U \subset \mathbb{C}$) WE HAVE A CONVERSE TO THE ABOVE: ALL MEROMORPHIC FUNCTIONS ON $U$ ARE RATIOS of HOLOMORPHIC FCNS. THIS IS BEYOND THE TECHNIQUES of THIS MODULE

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## (4) PATCHING, FACTORING.

**LEMMA**: SUPPOSE $f: U \to \hat{\mathbb{C}}$, $g: V \to \hat{\mathbb{C}}$ MEROMORPHIC. SUPPOSE $f|_{U \cap V} = g|_{U \cap V}$ (AND $U \cap V \neq \emptyset$). THEN THERE IS A UNIQUE MERO. EXTENSION of $f, g$ TO $h: U \cup V \to \hat{\mathbb{C}}$.

**PROOF**: PROVE $P(f) \cup P(g)$ ISOLATED IN $U \cup V$
$Z(f) \cup Z(g)$ " " $U \cup V$.
NOW PATCH $f|_{U - P(f)}$ WITH $g|_{V - P(g)}$. ∎

**LEMMA**: SUPPOSE $h: U \to \hat{\mathbb{C}}$ IS MEROMORPHIC. SUPPOSE $z_0 \in U$. SET $N = \mathrm{ORD}(h, z_0)$. THEN HAVE $H: U \to \hat{\mathbb{C}}$

MEROMORPHIC SO THAT $H(z_0) \neq 0, \infty$ AND

$$h(z) = (z - z_0)^N H(z_0) \quad \text{IN } U.$$

**PROOF**: APPLY REMOVABLE SING. THM TO $(z-z_0)^{-N+1} h(z)$.
NOW APPLY HOLOMORPHIC FACTORING. $\square$