2025-11-19 LECTURE 19 SAUL SCHLEIMER MA3B8

① MORE MEROMORPHICNESS

COROLLARY of FACTORING SUPPOSE $h : U \to \hat{\mathbb{C}}$ MERO.
THEN $Z(h) \cup P(h)$ ISOLATED IN U.

EXAMPLE: $\tan(z) = \sin(z)/\cos(z)$:
ZEROS DO NOT ACCUMULATE ON POLES, POLES DO NOT ACCUMULATE ON ZEROS.

CAREFUL: $\tan(1/z)$: ZEROS AND POLES ACCUMULATE ON THE ESSENTIAL SINGULARITY.

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② RESIDUES:

THE RESIDUE LEMMA: SUPPOSE $h : U \to \hat{\mathbb{C}}$ MEROMORPHIC.
SUPPOSE $z_0 \in P(h)$. FIX $\varepsilon > 0$ SO THAT $C(z_0; \varepsilon) \cap P(h) = \emptyset$
$B(z_0, \varepsilon) \cap P(h) = \{z_0\}$

THEN $\displaystyle\int_C h \, dz = 2\pi i \cdot \mathrm{RES}(h, z_0)$ □

THE RESIDUE THM: SUPPOSE $h : U \to \hat{\mathbb{C}}$ MEROMORPHIC.
SUPPOSE $R \subset U$ IS A REGION. SUPPOSE $\partial R \cap P(h) = \emptyset$.

THEN: $$\int_{\partial R} h \, dz = 2\pi i \sum_{w \in P(h) \cap R} \mathrm{RES}(h, w)$$

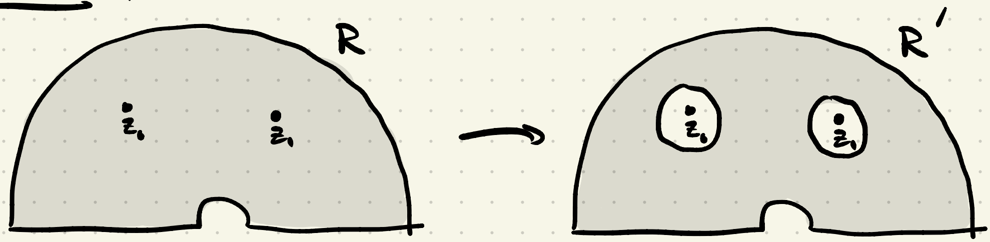
PROOF: R IS COMPACT, $P(h)$ IS ISOLATED (IN U)
SO $R \cap P(h)$ IS FINITE. FOR EACH $z_k \in P(h) \cap R$
PICK ε_k AND SET $B_k = B(z_k; \varepsilon_k)$ WITH

① $\overline{B_k} \subset U$

② $\overline{B_k} \cap B_\ell = \emptyset$ IFF $k \neq \ell$ ③ $C_k = \partial B_k$ DISJOINT FROM ∂R

SET $C_k = \partial B_k = C(z_k; \varepsilon_k)$. LET R' BE THE REGION $R - \cup B_k$

PICTURE:



NOTE R' IS AGAIN A REGION. SO BY CAUCHY (FOR REGIONS) $\int_{\partial R'} h\,dz = 0$.

RESIDUE LEMMA.

SO $\int_R h\,dz = \sum_k \int_{C_k} h\,dz = 2\pi i \sum_k \mathrm{RES}(h, z_k)$ $\square$

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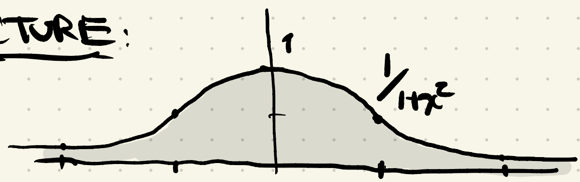
③ IMPROPER DEFINITE INTEGRALS.

EXAMPLE: $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R,S\to\infty} \int_{-R}^{S} \frac{dx}{1+x^2}$ CONVERGES.

NOW: $x = \tan(\theta)$, $dx = \sec^2\theta\, d\theta$, $1+\tan^2\theta = \sec^2\theta$

SO $\int_{-R}^{S} \frac{dx}{1+x^2} = \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \theta \Big] = \arctan(x)\Big]_{-R}^{S} \to \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi = I}$

PICTURE:



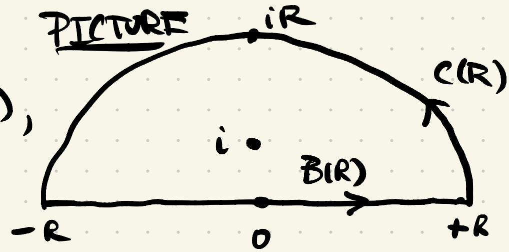
ANOTHER WAY: DEFINE $f(z) = \frac{1}{1+z^2}$. SO $P(f) = \{\pm i\}$

ARE SIMPLE POLES.

$$\mathrm{RES}(f, i) = \lim_{z\to i} \frac{z-i}{1+z^2} = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}.$$

TAKE $R>1$, $B(R)=[-R,R]$, $C(R)=$ UPPER HALF of $C(0;R)$, AND $A(R)=B(R)\cup C(R)$.

PICTURE



SO $\int_{A(R)} f\,dz = 2\pi i \,\mathrm{RES}(f,i) = \pi$ BY RES. THM.

NOTE $\int_{B(R)} f\,dz \xrightarrow[R\to\infty]{} I$. SO: IT SUFFICES TO PROVE $\left|\int_{C(R)} f\,dz\right| \xrightarrow[R\to\infty]{} 0$. WE COMPUTE:

$$\left|\int_{C(R)} f\,dz\right| = \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{1+R^2e^{2i\theta}}\right|$$

$$\leq \int_0^{\pi} \frac{R\,d\theta}{R^2-1}$$

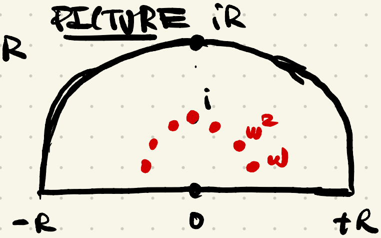
$$\leq \frac{R}{R^2-1}\cdot\pi \xrightarrow[R\to\infty]{} 0$$

□

4 GOING THROUGH A PHASE

DEFINE $I_n = \int_0^{\infty} \frac{dx}{1+x^n}$. THIS CONVERGES. IF n EVEN WE CAN REUSE THE HALF CIRCLE CONTOUR. ALL POLES ARE SIMPLE SO SUM THE RESIDUES AND WIN. [DON'T FORGET TO DIVIDE BY TWO!].

PICTURE

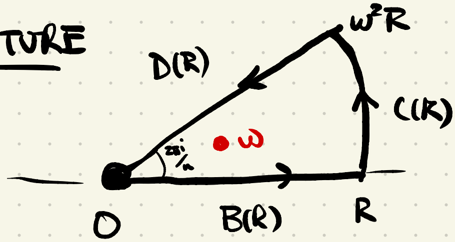


HERE IS ANOTHER WAY:

DEFINE $w=\mathrm{EXP}(\pi i/n)$. THIS IS A $2n^{th}$ ROOT, SO SOLVES $1-x^{2n}=(1+x^n)(1-x^n)$.

LET $B(R) = [0,R]$, $C(R) =$ ARC of $C(0;R)$ FROM R TO $\omega^2 R$,
$D(R) = [R,0]\cdot\omega^2$.

PICTURE



$A(R) = B(R) \cup C(R) \cup D(R)$ IS A WEDGE CONTOUR.

SET $f(z) = \dfrac{1}{1+z^n}$.

SO $\lim\limits_{R\to\infty} \int_{B(R)} f\,dz = I_n$. EXERCISE: $\int_{C(R)} f\,dz \xrightarrow[R\to\infty]{} 0$.

WE PARAMETRISE $D(R)$ BY $\gamma : [0,R] \longrightarrow \mathbb{C}$
$t \longmapsto t\cdot\omega^2$.] NOTE ORIENTATION.

$$\text{SO } \int_{D(R)} f\,dz = \int_0^R \frac{\omega^2 dt}{1+\omega^{2n}t^n} = \omega^2 \int_0^R \frac{dt}{1+t^n} \xrightarrow[R\to\infty]{} \omega^2 I_n.$$

$$\text{SO } \int_{A(R)} f\,dz \xrightarrow[R\to\infty]{} I_n - \omega^2 I_n = (1-\omega^2) I_n.$$

ALSO: $\operatorname{RES}(f,\omega) = \lim\limits_{z\to\omega} \dfrac{z-\omega}{1+z^n} = \lim\limits_{z\to\omega} \dfrac{1}{nz^{n-1}} = \dfrac{1}{n\omega^{n-1}}$

$= \dfrac{\omega^{n+1}}{n}$

SO: $(1-\omega^2) I_n = 2\pi i \dfrac{\omega^{n+1}}{n}$

$$\text{So } I_n = \frac{2\pi i}{n} \frac{\omega^n}{\frac{1}{\omega}-\omega} = \frac{2\pi i}{n} \frac{+1}{\omega - \frac{1}{\omega}}$$

$$= \frac{2\pi i}{n} \frac{1}{\cos(\pi/n) + i\sin(\pi/n) - \cos(\pi/n) + i\sin(\pi/n)}$$

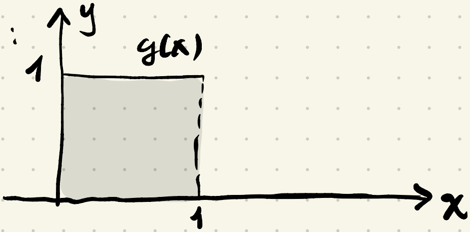
$$= \frac{2\pi i}{n} \frac{1}{2i\sin(\pi/n)} = \frac{\pi/n}{\sin(\pi/n)}.$$

□

SANITY CHECK : NOTE THAT $\dfrac{\pi/n}{\sin(\pi/n)} \longrightarrow 1$ AS $n \to \infty$

THE GRAPH of $1/_{1+x^n}$ CONVERGES TO $g(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \in (1,\infty) \end{cases}$

PICTURE :



SO $I_n \longrightarrow \int_0^\infty g(x)\,dx = 1$

AS HOPED FOR. $\square$