2025-11-21 LECTURE 20 SAULSCHLEIMER MA 3B8

① TRIGONOMETRIC INTEGRALS

$$I_S = \int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2}\,dx \qquad I_c = \int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx$$

STEPS: (i) CHECK CONVERGENCE.

(ii) LOOK FOR SIMPLIFICATIONS.

(iii) PICK FUNCTION AND CONTOUR.

(iv) COMPUTE RESIDUES.

(v) ESTIMATE AWAY "MINOR ARCS".

(vi) CHECK WORK.

(i) $|\sin(x)|, |\cos(x)| \leq 1$ FOR ALL x SO

$$|I_S|, |I_c| \leq \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi. \text{ SO CONVERGES}$$

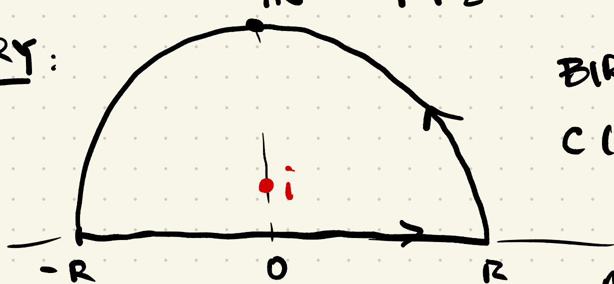
(ii) $\frac{1}{1+x^2}$ EVEN SO $\frac{\sin(x)}{1+x^2}$ ODD, SO $I_S = 0$.

FOR I_c: $\cos(x) = \text{REAL}(\text{EXP}(ix))$

SO $I_c = \text{REAL}\left(\int_{-\infty}^{\infty} \frac{\text{EXP}(ix)}{1+x^2}\,dx\right)$, $I_S = \text{IMAG}\left(\int_{-\infty}^{\infty} \frac{\text{EXP}(ix)}{1+x^2}\,dx\right)$

SO DEFINE $f(z) = \frac{\text{EXP}(iz)}{1+z^2}$. HAS SIMPLE POLES AT $\pm i$.

(iii) TRY:



$B(R) = [-R, R]$

$C(R) =$ UPPER HALF of $C(0;R)$.

$A(R) = B(R) \cup C(R)$.

(iv) $\text{RES}(f, i) = \lim_{z \to i} (z-i) \cdot \frac{\text{EXP}(iz)}{1+z^2}$

$$= \frac{\text{EXP}(-1)}{2i}$$

$$= \frac{1}{2ie}.$$

RESIDUE THM: $\int_{A(R)} f\,dz = 2\pi i \cdot \frac{1}{2ie} = \frac{\pi}{e}.$

NOTE $\int_{B(R)} f\,dz \longrightarrow I_c$ AS $R \to \infty$. SO, SUFFICES TO BOUND $\int_{C(R)} f\,dz$.

(v) WE COMPUTE

$$\int_{C(R)} f\,dz = \int_0^\pi \frac{\text{EXP}(iRe^{i\theta})}{1+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta$$

$$= \int_0^\pi \frac{\text{EXP}(iR\cos(\theta) - R\sin(\theta))}{1+R^2e^{2i\theta}} iRe^{i\theta}\,d\theta$$

$$= \int_0^\pi \frac{\underline{\text{EXP}(iR\cos\theta)} \cdot \underline{\text{EXP}(-R\sin\theta)}}{1+R^2e^{2i\theta}} \underline{i}R\underline{e^{i\theta}}\,d\theta$$

SO: $\left| \int_{C(R)} f\,dz \right| \leq \int_0^\pi \frac{R}{R^2-1}\,d\theta = \frac{R}{R^2-1} \cdot \pi \xrightarrow[R \to \infty]{} 0.$

(vi) SANITY CHECK: USE COMPUTER! $\square$

② **HOMEOMORPHISM** :

DEF: SUPPOSE X, Y TOPOLOGICAL SPACES. SUPPOSE $f: X \to Y$, $g: Y \to X$ CONTINUOUS AND $f \circ g = ID_Y$, $g \circ f = ID_X$. THEN CALL $f: X \to Y$ A **HOMEOMORPHISM** AND CALL X, Y HOMEOMORPHIC.

EX: $\mathbb{D}$ HOMEO TO $\mathbb{C}$.

INVARIANCE: SUPPOSE $f: X \to Y$ HOMEOMORPHISM, WITH INVERSE $g: Y \to X$. THEN f, g INDUCE ISOMORPHISMS:

$$\begin{tikzcd} H_n(X) \arrow[r, bend left, "{[f_n]}"] & H_n(Y) \arrow[l, bend left, "{[g_n]}"] \end{tikzcd}$$

WITH $[f_n] \circ [g_n] = ID_{H_n(Y)}$

$[g_n] \circ [f_n] = ID_{H_n(X)}$.

WE ONLY NEED THIS FOR $n = 1$.

PROOF: f INDUCES
$$f_n : C_n(X) \longrightarrow C_n(Y)$$
$$\sigma^n \longmapsto f \circ \sigma^n$$
WHICH COMMUTES WITH $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$.
SO PRESERVES CYCLES, BOUNDARIES, ETC. $\square$.

RECALL: IF U CONVEX THEN $H_1(U) \cong 0$. [CONING!]
SO IF U HOMEO TO V THEN $H_1(V) \cong 0$ AS WELL.

EXERCISE $\mathbb{C}$ IS NOT HOMEO TO $\mathbb{C}^*$.
[$H_1(\mathbb{C}) \cong 0$; BUT $H_1(\mathbb{C}^*) \not\cong 0$].

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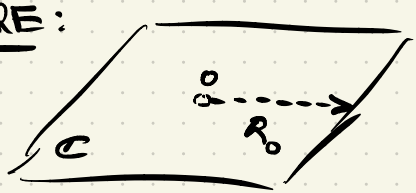
③ **BRANCH CUTS**: FIX AN ANGLE θ. DEFINE
$R_\theta = \{ t e^{i\theta} \mid t \in [0, \infty) \}$ = RAY of ANGLE θ.

DEFINE $C_\theta = \mathbb{C} - R_\theta$ TO BE THE CUT PLANE

[ALSO CALL R_θ A BRANCH CUT.]

PICTURE:



DEFINE

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \text{IMAG}(z) > 0 \}$$

THEN

$$\mathbb{H} \xrightarrow{\cong} C_0$$
$$z \longmapsto z^2$$

IS A HOMEOMORPHISM.

SINCE $\mathbb{H}$ CONVEX, $H_1(\mathbb{H}) \cong 0$. THUS $H_1(C_0) \cong 0$. ∎