2025-12-02 LECTURE 24 SAUL SCHLEIMER MA3B8

ADMIN: (1) ALL LECTURE NOTES POSTED.
PLEASE SEND ME CORRECTIONS!
(2) THE NOTES HAVE WORKED EXERCISES.
(3) OFFICE HOURS 11-2 WEDNES IN B2.14.
(3) FINAL MODULE EVALUATION FORM IS UP.

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(1) CONFORMAL EQUIVALENCE: OUR FINAL TOPIC.

DEF: SUPPOSE U, V DOMAINS. SUPPOSE $f: U \to V$ HOLOMORPHIC WITH HOLOMORPHIC INVERSE. THEN CALL f **BIHOLOMORPHIC** OR **CONFORMAL EQUIVALENCE**. CALL U, V **CONFORMALLY EQUIVALENT**.

LEMMA: IF U, V CONF. EQUIV. THEN U, V HOMEOMORPHIC.
HOWEVER THE CONVERSE IS (VERY!) FALSE.

EXAMPLE: $\mathbb{C}, \mathbb{D}$ ARE HOMEOMORPHIC BUT NOT CONF. EQUIV.

PF: DEFINE $F: \mathbb{C} \to \mathbb{D}$ BY $F(re^{i\theta}) = \frac{r}{1+r} e^{i\theta}$. THIS IS A HOMEO. BUT IF $f: \mathbb{C} \to \mathbb{D}$ HOLOMORPHIC THEN f IS CONSTANT. [Liouville] $\square$

EXERCISE: $\mathbb{C}^\times$ AND $\mathbb{D}^\times$ HOMEO BUT NOT CONF. EQUIV.

EXERCISE: SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC, INJECTIVE. THEN $f: U \to f(U)$ IS BIHOLOMORPHIC [HAS INVERSE].

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(2) AUTOMORPHISMS:

DEFINE: SUPPOSE U A DOMAIN. THEN
$$\mathrm{AUT}(U) = \{ f: U \to U \mid f \text{ BIHOLOMORPHIC} \}.$$

SO $\mathrm{AUT}(U)$ IS A GROUP (FUNCTION COMPOSITION IS THE GROUP OPERATION)

RECALL: $\mathrm{SIM}(\mathbb{C})$ IS THE GROUP of SIMILARITIES: FUNCTIONS $f: \mathbb{C} \to \mathbb{C}$ WITH $f(z) = az + b$, $a \in \mathbb{C}^{\times}$, $b \in \mathbb{C}$.

THEOREM: $\mathrm{AUT}(\mathbb{C}) = \mathrm{SIM}(\mathbb{C})$.

PROOF: IF $f(z) = az + b$ THEN $g(z) = \frac{1}{a} z - \frac{b}{a}$ IS THE INVERSE. SO SIMILARITIES ARE AUTOMORP.

SUPPOSE $h \in \mathrm{AUT}(\mathbb{C})$. SINCE $\mathbb{D} \cap \mathbb{C} - \overline{\mathbb{D}} = \emptyset$ WE HAVE $h(\mathbb{D}) \cap h(\mathbb{C} - \overline{\mathbb{D}}) = \emptyset$. NOTE h IS A HOMEOMORPHISM. SO $h(\mathbb{D})$ OPEN. SO $h(\mathbb{C} - \overline{\mathbb{D}})$ NOT DENSE IN $\mathbb{C}$. SO h DOES NOT HAVE AN ESS. SING AT ∞. [CASORATI-WEIERSTRASS].

NOTE h IS ENTIRE SO BY CAUCHY, h HAS SERIES CONVERGING IN $\mathbb{C}$. SINCE h HAS NO ESS. SING AT ∞, h IS A POLYNOMIAL. SINCE h IS BIJECTIVE, [FUND THM of ALGEBRA] h IS LINEAR. $\square$

③ TRANSITIVITY:

SUPPOSE G A GROUP ACTS ON X A SET. THE ACTION IS TRANSITIVE IF FOR ALL $x, y \in X$, THERE IS SOME $g \in G$ SO THAT $g \cdot x = y$.

SAY ACTION IS k-TRANSITIVE IF FOR ANY k-TUPLES $(x_1, \cdots, x_k)$ AND $(y_1, \cdots, y_k)$ of DISTINCT ELTS of X WE HAVE SOME $g \in G$ SO THAT $g \cdot x_i = y_i$ FOR ALL i.

LEMMA: THE ACTION of $SIM(\mathbb{C})$ ON $\mathbb{C}$ IS (UNIQUELY) TWO-TRANSITIVE.

PROOF: SUPPOSE $p, q \in \mathbb{C}$ WITH $p \neq q$. WE SOLVE

$$\left.\begin{aligned} f(p) &= 0 = ap + b \\ f(q) &= 1 = aq + b \end{aligned}\right\} \begin{aligned} &\text{SO } b = -ap \\ &\text{SO } 1 = aq - ap \\ &\text{SO } a = 1/q-p. \end{aligned}$$

SO: $f(z) = \dfrac{z-p}{q-p} = \dfrac{1}{q-p} \cdot z - \dfrac{p}{q-p} \in SIM(\mathbb{C})$

AND f IS ONLY SUCH. NOW APPEAL TO AXIOMS of GROUP ACTION. $\square$

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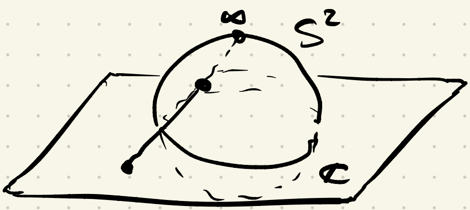
④ THE EXTENDED PLANE:

WE DEFINE $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. WE TOPOLOGISE $\hat{\mathbb{C}}$ BY TAKING $U \subset \hat{\mathbb{C}}$ OPEN IFF

(i) $\infty \notin U$ AND U OPEN IN $\mathbb{C}$ OR

(ii) $\infty \in U$ AND $K = \hat{\mathbb{C}} - U$ COMPACT IN $\mathbb{C}$.

PICTURE:



EXERCISE: $\hat{\mathbb{C}}$ IS COMPACT, $\hat{\mathbb{C}}$ IS HOMEO TO S^2.

EXAMPLE: SUPPOSE L, M ARE LINES IN $\mathbb{C}$.

DEFINE $\hat{L}$ = CLOSURE of L IN $\hat{\mathbb{C}} = L \cup \{\infty\}$.

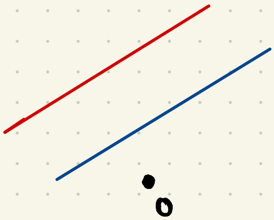
$\hat{M}$ = " M " " " $M \cup \{\infty\}$.

CLAIM (i) IF L, M PARALLEL THEN $\hat{L}, \hat{M}$ TANGENT AT ∞.

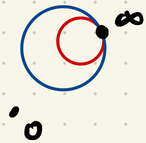
(ii) IF L, M CROSS W/ANGLE θ THEN $\hat{L}, \hat{M}$ CROSS W/ANGLE θ AT INFINITY.

PICTURES

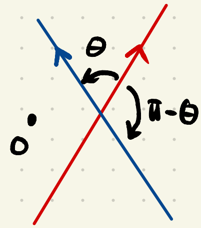
IN $\mathbb{C}$



IN $\hat{\mathbb{C}}$



IN $\mathbb{C}$



IN $\hat{\mathbb{C}}$