2025-12-03 LECTURE 25 SAUL SCHLEIMER MA3B8

~40

① ALGEBRA TO GEOMETRY

RECALL $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. FIX $w \in \mathbb{C}^\times$.

DEFINE: $A_w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\quad A_w(z) = \begin{cases} z + w & z \neq \infty \\ \infty & z = \infty \end{cases}$

$M_w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\quad M_w(z) = \begin{cases} w \cdot z, & z \neq \infty \\ \infty, & z = \infty \end{cases}$

$V : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\quad V(z) = \begin{cases} 1/z & z \neq 0, \infty \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$

EXERCISE: A_w, M_w, V HOMEOMORPHISMS of $\hat{\mathbb{C}}$.

DEF: A FUNCTION $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ IS MEROMORPHIC IF IT IS MEROMORPHIC ON $\mathbb{C}$ AND, FOR ANY $r > 0$, $(f \circ V)|B(0;r)$ IS MEROMORPHIC.

EXERCISE A_w, M_w, V ARE BIMEROMORPHIC.

PROOF: $A_w \circ V = w/z$ IS MEROMORPHIC AT ZERO. ETC. □

DEFINE: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ IS A CONFORMAL EQUIV (AUTOMORPHISM) IF f IS BIMEROMORPHIC.

SO: $f(z) = \dfrac{z+i}{iz+1}$ IS BIMEROMORPHIC ON $\hat{\mathbb{C}}$.

$f(z) = \exp(z)$ IS NOT [ESS. SING. AT ∞!]

DEFINE: $\mathrm{AUT}(\hat{\mathbb{C}}) = \{ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f \text{ BIMEROMORPHIC} \}$

SO $A_w, M_w, V \in \mathrm{AUT}(\hat{\mathbb{C}})$. BUT $z \mapsto z^2$ IS NOT.

② <u>LEMMA</u>: SUPPOSE $a,b,c,d \in \mathbb{C}$ WITH $ad-bc \neq 0$. THEN $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ DEFINED BY $f(z) = \dfrac{az+b}{cz+d}$ IS AN AUTOMORPHISM of $\hat{\mathbb{C}}$. FURTHERMORE ALL $h \in \mathrm{AUT}(\hat{\mathbb{C}})$ HAVE THIS FORM.

<u>PROOF</u>: $f(z) = (az+b)/(cz+d)$ HAS INVERSE

$$g(z) = (dz-b)/(-cz+a) \quad \text{[CHECK THIS!]}$$

ALSO $f|\mathbb{C}$ MEROMORPHIC AS IS

$$f \circ V(z) = \frac{a/z+b}{c/z+d} = \frac{bz+a}{dz+c}$$

SUPPOSE $f \in \mathrm{AUT}(\hat{\mathbb{C}})$. SUPPOSE $f(\infty) = \infty$. THEN $f|\mathbb{C}$ IS IN $\mathrm{AUT}(\mathbb{C})$, SO $f \in \mathrm{SIM}(\mathbb{C})$ SO

$$f(z) = az+b = \frac{az+b}{0z+1}.$$

SUPPOSE $f(\infty) = p \neq \infty$. DEFINE $g(z) = \dfrac{1}{z-p}$. SO $g \circ f \in \mathrm{AUT}(\hat{\mathbb{C}})$ AND $g \circ f$ FIXES ∞. SO $g \circ f(z) = cz+d$ WITH $c \neq 0$. SO

$$\frac{1}{f(z)-p} = cz+d \quad \text{THUS} \quad \frac{1}{cz+d} = f(z) - p$$

$$\text{SO} \quad f(z) = p + \frac{1}{cz+d} = \frac{pcz+(pd+1)}{cz+d}$$

NOTE $(pc \cdot d) - (pd+1)(c) = pcd - pcd - c = -c \neq 0$.

SO f HAS THE DESIRED FORM $\square$

③ THREE-TRANSITIVE

LEMMA: THE ACTION OF $\mathrm{AUT}(\hat{\mathbb{C}})$ ON $\hat{\mathbb{C}}$ IS (UNIQUELY) THREE-TRANSITIVE.

PROOF: FIX $u, v, w \in \hat{\mathbb{C}}$ DISTINCT. WE WILL PROVE: THERE IS A **UNIQUE** $f \in \mathrm{AUT}(\hat{\mathbb{C}})$ SENDING (u, v, w) TO $(\infty, 0, 1)$ [IN THAT ORDER].

(A) SUPPOSE $u, v, w \neq \infty$. SO $f(z) = \dfrac{w-u}{w-v} \cdot \dfrac{z-v}{z-u}$

(B) SUPPOSE $u = \infty$. SO $f(z) = \dfrac{z-v}{w-v}$

(C) SUPPOSE $v = \infty$ SO $f(z) = \dfrac{w-u}{z-u}$

(D) SUPPOSE $w = \infty$ SO $f(z) = \dfrac{z-v}{z-u}$. $\square$

SKIP!

EXERCISE: LIST ALL $f \in \mathrm{AUT}(\hat{\mathbb{C}})$ FIXING $\{\infty, 0, 1\}$ SETWISE.
[**HINT**: BY THE ABOVE THERE ARE SIX SUCH]

③ $\mathrm{AUT}(\mathbb{C}^\times)$:

LEMMA $\mathrm{AUT}(\mathbb{C}^\times) = \{ M_w, M_w \circ V \mid w \in \mathbb{C}^\times \}$

IN FACT: $\mathrm{AUT}(\mathbb{C}^\times) \cong \mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$

$$\left[(V \circ M_w \circ V)(z) = \frac{1}{w/z} = z/w = M_{1/w}(z) \right]$$

IN FACT: $\mathrm{AUT}(\mathbb{C}^\times) \cong \{ f \in \mathrm{AUT}(\hat{\mathbb{C}}) \mid f(\{0,\infty\}) = \{0,\infty\} \}$

④ GL(2, $\mathbb{C}$): WE WILL GIVE A DICTIONARY BETWEEN LFT (LINEAR FRACTIONAL TRANSFORMATIONS) AND MATRICES. HERE ARE A FEW PAGES.

LFT	MATRIX
$A_b(z) = z + b$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$
$M_{a^2}(z) = a^2 z$	$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$
$Y(z) = 1/z$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$f(z) = \dfrac{az+b}{cz+d}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

WE START BY DEFINING LOTS of GROUPS.

⑤ LINEAR GROUPS.

(1) $GL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{c} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{array} \right\}.$

GENERAL LINEAR GROUP.

(2) $SL(2,\mathbb{C}) = \{ M \in GL(2,\mathbb{C}) \mid \mathrm{DET}(M) = 1 \}.$

SPECIAL LINEAR GROUP.

(3) $\mathbb{C}^\times \cdot \mathrm{Id} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}^\times \right\}$

(4) $PGL(2,\mathbb{C}) = GL(2,\mathbb{C}) / \mathbb{C}^\times \cdot \mathrm{Id}$

$PSL(2,\mathbb{C}) = SL(2,\mathbb{C}) / \pm \mathrm{Id}.$

} PROJECTIVISATIONS of GL, SL.

AND: CAN MAKE SAME DEFS OVER $\mathbb{R}$.

[BUT $PGL(2,\mathbb{R}) = GL(2,\mathbb{R}) / \mathbb{R}^\times \cdot \mathrm{Id}$]

SO MANY HOMOMORPHISMS:

$$\begin{tikzcd}
& \mathbb{C}^\times\cdot \mathrm{Id} \arrow[rr] & & GL(2,\mathbb{C}) \arrow[rr] & & PGL(2,\mathbb{C}) \\
\mathbb{R}^\times\cdot\mathrm{Id} \arrow[ur] \arrow[rr] & & GL(2,\mathbb{R}) \arrow[ur] \arrow[rr] & & PGL(2,\mathbb{R}) \arrow[ur] & \\
& \{\pm \mathrm{Id}\} \arrow[uu] \arrow[rr] & & SL(2,\mathbb{C}) \arrow[uu] \arrow[rr] & & PSL(2,\mathbb{C}) \arrow[uu] \\
\{\pm\mathrm{Id}\} \arrow[uu] \arrow[ur] \arrow[rr] & & SL(2,\mathbb{R}) \arrow[uu] \arrow[ur] \arrow[rr] & & PSL(2,\mathbb{R}) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

LEMMA THE INDUCED HOMOMORPHISMS HAVE

$\rho_{\mathbb{C}} : PSL(2,\mathbb{C}) \xrightarrow{\cong} PGL(2,\mathbb{C})$ $\quad \rho_{\mathbb{R}} : PSL(2,\mathbb{R}) \xrightarrow[\text{TWO}]{\text{INDEX}} PGL(2,\mathbb{R})$