2025-12-05 LECTURE 26 MA3B8 SAUL SCHLEIMER

(*) FINAL MODULE EVALUATION FORM

(1) UNITARY GROUPS

DEFINE $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

SIGNATURE $\quad (1,1) \quad (2,0) \quad (0,2)$

[COUNTS NUMBER of ± 1 EIGENVALUES]

DEFINE FOR $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ DEFINE $M^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

= CONJUGATE TRANSPOSE.

DEFINE $U(1,1) = \{ M \in GL(2,\mathbb{C}) \mid M^* J M = J \}$ UNITARY GROUP

$SU(1,1) = \{ M \in U(1,1) \mid \det(M) = 1 \}$ SPECIAL UNITARY

$\left.\begin{array}{l} PU(1,1) = U(1,1) / S^1 \cdot Id \\ PSU(1,1) = SU(1,1) / \pm Id \end{array}\right\}$ PROJECTIVISATIONS.

[CAN MAKE SAME DEFS FOR $U(2) = U(2,0)$. BUT LESS IMPORTANT FOR US. LIKEWISE $U_{\mathbb{R}}(1,1)$.]

DIAGRAM:

$$\begin{tikzcd}
S^1 \cdot Id \arrow[r] & U(1,1) \arrow[r] & PU(1,1) \\
\pm Id \arrow[r] \arrow[u] & SU(1,1) \arrow[r] \arrow[u] & PSU(1,1) \arrow[u]
\end{tikzcd}$$

(2) MANY EXERCISES.

(i) PROVE $U(1,1)$ IS A GROUP

(ii) PROVE $PSL(2,\mathbb{C}) \cong PGL(2,\mathbb{C})$.

(iii) PROVE $PSL(2,\mathbb{R}) < PGL(2,\mathbb{R})$ INDEX TWO.

(iv) PROVE

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \;\middle|\; \begin{matrix} a, b \in \mathbb{C} \\ |a|^2 - |b|^2 = 1 \end{matrix} \right\}$$

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③ SUBGROUPS of $AUT(\hat{\mathbb{C}})$.

SUPPOSE THAT U IS A DOMAIN (OR $U = \hat{\mathbb{C}}$). SUPPOSE $X \subset U$.

DEFINE $AUT_X(U) = \{ f \in AUT(U) \mid f(X) = X \}$.

THIS IS THE SETWISE STABILISER of X IN $AUT(U)$.

IF $X = \{p\}$ IS A SINGLETON. THEN INSTEAD WRITE $AUT_p(U)$.

EXAMPLE $AUT_\infty(\hat{\mathbb{C}}) \cong SIM(\mathbb{C})$.

EXAMPLE: $AUT_{\{0,\infty\}}(\hat{\mathbb{C}}) \cong \mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$

$$= \{ a \cdot z, \; a/z \mid a \in \mathbb{C}^\times \}.$$

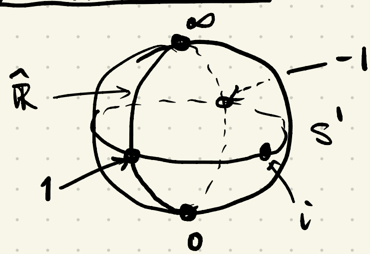
DEFINE: $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ THE EXTENDED LINE

EXERCISE $\hat{\mathbb{R}}$ IS HOMEO TO S^1.

IN FACT,

$$f(z) = \frac{iz + 1}{z + i}$$

TAKES $\hat{\mathbb{R}}$ TO S^1. [CHECK].



DEFINE: $\rho : PGL(2,\mathbb{C}) \longrightarrow AUT(\hat{\mathbb{C}})$

$$[A] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \longmapsto f_A(z) = \frac{az+b}{cz+d}.$$

WE PROVED THIS IS AN ISOMORPHISM.

EXERCISES

(1) $\rho(PGL(2,\mathbb{R})) = AUT_{\hat{\mathbb{R}}}(\hat{\mathbb{C}})$

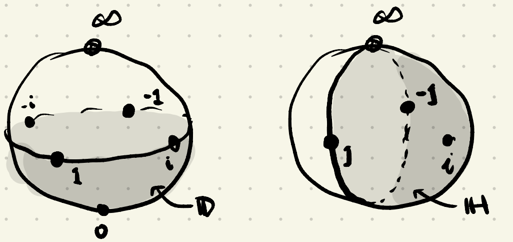
(2) $\rho(PSL(2,\mathbb{R})) = AUT_{\mathbb{H}}(\hat{\mathbb{C}})$ $\quad \left[\mathbb{H} = \{z \in \mathbb{C} \mid IMAG(z) > 0\}\right]$

(3) $\rho(PU(1,1)) = AUT_{S^1}(\hat{\mathbb{C}})$

(4) $\rho(PSU(1,1)) = AUT_{\mathbb{D}}(\hat{\mathbb{C}})$ $\quad \left[\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}\right]$

ALSO $f(z) = \dfrac{z+i}{iz+1}$ SENDS $0 \longmapsto i$, $i \longmapsto \infty$, $\infty \longmapsto -i$, $-i \longmapsto 0$ AND $1 \longmapsto 1$, $-1 \longmapsto -1$

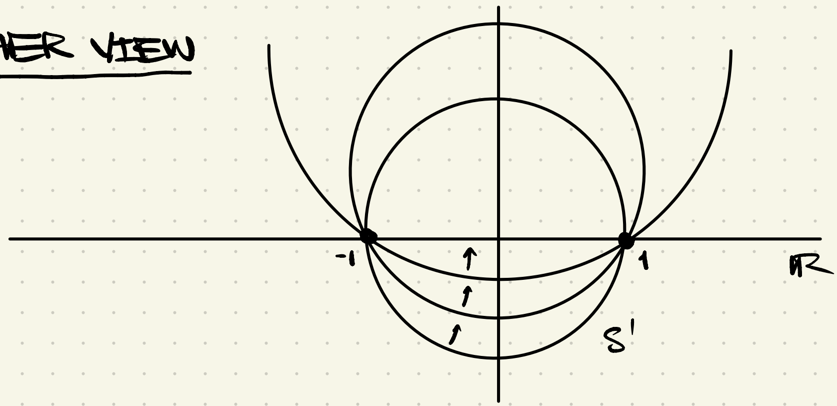
PICTURE: IN $\hat{\mathbb{C}}$



SO $f(\mathbb{D}) = \mathbb{H}$

SO $\mathbb{D}$ CONFORM. EQUIV. TO $\mathbb{H}$.

ANOTHER VIEW



IN $\mathbb{C}$

EXERCISE: TAKE $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $f = f_A = \rho([A])$.

THEN: (a) $f(S^1) = \hat{\mathbb{R}}$, $f(\mathbb{D}) = \mathbb{H}$

(b) $AUT_{S^1}(\hat{\mathbb{C}}) = f^{-1} \circ AUT_{\hat{\mathbb{R}}}(\hat{\mathbb{C}}) \circ f$

(c) $AUT_{\mathbb{D}}(\hat{\mathbb{C}}) = f^{-1} \circ AUT_{\mathbb{H}}(\hat{\mathbb{C}}) \circ f$

(d) $PU(1,1) = A^{-1} \circ PGL(2,\mathbb{R}) \circ A$

(e) $PSU(1,1) = A^{-1} \circ PSL(2,\mathbb{R}) \circ A$

<u>DIAGRAM</u>:

$$
\begin{tikzcd}
 & PSL(2,\mathbb{R}) \arrow[rr, hook] \arrow[dl, "p"'] \arrow[dd, squiggly, "\text{CONJ BY } A"] & & PGL(2,\mathbb{R}) \arrow[dl, "p"'] \arrow[dd, squiggly, "\text{CONJ BY } A"] \\
\mathrm{AUT}_{\mathbb{H}}(\hat{\mathbb{C}}) \arrow[rr, hook] \arrow[dd, squiggly, "\text{CONJ BY } f"] & & \mathrm{AUT}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}}) \arrow[dd, squiggly, "\text{CONJ BY } f"] & \\
 & PSU(1,1) \arrow[rr, hook] \arrow[dl, "p"'] & & PU(1,1) \arrow[dl, "p"'] \\
\mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}}) \arrow[rr, hook] & & \mathrm{AUT}_{S^1}(\hat{\mathbb{C}}) &
\end{tikzcd}
$$

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(4) <u>SCHWARZ'S LEMMA</u>

<u>LEMMA</u>: SUPPOSE $f: \mathbb{D} \to \mathbb{D}$ HOLOMORPHIC AND $f(0)=0$.

THEN (i) $|f(z)| < |z|$ FOR ALL $z \in \mathbb{D}$.

AND (ii) $|f'(0)| \leq 1$

MOREOVER IF (i) $|f(z)| = |z|$ FOR ANY $z \in \mathbb{D}$

OR (ii) $|f'(0)| = 1$

<u>THEN</u> f IS A ROTATION.

<u>PROOF</u>: IF f CONSTANT THEN $f \equiv 0$ AND DONE. SO SUPPOSE f NON CONSTANT. SINCE $f(0)=0$ $\mathrm{ORD}(f,0) \geq 1$. FACTOR TO GET $g: \mathbb{D} \to \mathbb{C}$ SO THAT $f(z) = z \cdot g(z)$. SO

$$g(z) = \begin{cases} f(z)/z, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

FIX ANY $z_0 \in \mathbb{D}^\times$. FIX ANY $r > 0$ SO THAT $|z_0| < r < 1$.

WE APPLY THE MAX. MOD. PRINCIP. TO g ON $\overline{B(0;r)}$

WE FIND $|g(z_0)| \leq |g(z_r)|$

SO $\dfrac{|f(z_0)|}{|z_0|} = |g(z_0)| \le |g(z_r)| = \dfrac{|f(z_r)|}{|z_r|} \le \dfrac{1}{r}$

SO $|f(z_0)| \le \dfrac{|z_0|}{r}$ AS r TENDS TO 1 WE OBTAIN $|f(z_0)| \le |z_0|$.

NOTE $|f'(0)| = \lim_{z \to 0} \left|\dfrac{f(z)}{z}\right| = \lim_{z \to 0} |g(z)|$ IS THUS AT MOST ONE.

NOW SUPPOSE $|f(z_0)| = |z_0|$ OR $|f'(0)| = 1$. THEN MAX MOD of $|g|$ REALISED IN $\mathbb{D}$. SO g CONSTANT. SO $\dfrac{f(z)}{z} = a \in \mathbb{C}$.
SO $|a| = 1$ □.

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(5) $\underline{\mathrm{AUT}(\mathbb{D}) = \mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}}) = \rho(\mathrm{PSU}(1,1))}$

LEMMA: SUPPOSE $f(z) = \dfrac{az+b}{\bar{b}z+\bar{a}}$ WHERE $|a|^2 - |b|^2 = 1$. THEN $f \in \mathrm{AUT}(\mathbb{D})$. FURTHERMORE EVERY $h \in \mathrm{AUT}(\mathbb{D})$ IS of THIS FORM.

PROOF: $f \in \mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}})$ BY EXERCISES.

$\mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}}) \subset \mathrm{AUT}(\mathbb{D})$ BY RESTRICTION.

SUPPOSE $h \in \mathrm{AUT}(\mathbb{D})$. SUPPOSE $h(0) = 0$. LET $g = h^{-1}$ [ALSO IN $\mathrm{AUT}(\mathbb{D})$.]. SO $h \circ g = \mathrm{Id}_{\mathbb{D}}$ AND SO

$$h'(g(0)) \cdot g'(0) = 1$$

NOTE $g(0) = 0$. SO $|h'(0)| \cdot |g'(0)| = 1$. BY SCHWARZ $|h'(0)| \le 1$, $|g'(0)| \le 1$. SO $|h'(0)| = |g'(0)| = 1$. SO BOTH ROTATIONS. SUPPOSE $h(z) = \lambda^2 z$. SO $h(z) = \dfrac{\lambda z + 0}{0z + 1/\lambda}$
SINCE $1/\lambda = \bar{\lambda}$, h HAS DESIRED FORM.

SUPPOSE INSTEAD $h(0) = p$. DEFINE $g : \mathbb{D} \to \mathbb{D}$ BY

$$g(z) = \frac{z - p}{-\bar{p}z + 1}$$

NOTE $1 - |p|^2 > 0$. TAKE $\mu = \dfrac{1}{\sqrt{1 - |p|^2}}$ [REAL]

So $g(z) = \dfrac{\mu z - \mu p}{-\mu \bar{p} z + \mu}$ LIES IN $\rho(PSU(1,1))$.

NOW: $g \circ h : \mathbb{D} \to \mathbb{D}$ LIES IN $\mathrm{AUT}(\mathbb{D})$ AND FIXES ZERO.

SO $f = g \circ h$ IS A ROTATION, SAY $f(z) = \dfrac{\lambda z + 0}{0 + 1/\lambda} = \lambda^2 z$.

SO

$$h(z) = (g^{-1} \circ f)(z) = g^{-1}(\lambda^2 z) = \frac{\mu \lambda^2 z + \mu p}{\mu \bar{p} \lambda^2 z + \mu}$$

$$= \frac{\mu \lambda z + \mu \bar{\lambda} p}{\mu \bar{p} \lambda z + \mu \bar{\lambda}} \quad [\bar{\lambda} = 1/\lambda]$$

NOTE $\bar{\mu} = \mu \in \mathbb{R}$. SO $\overline{\mu\lambda} = \mu\bar{\lambda}$, $\overline{\mu\bar{\lambda}p} = \bar{\mu}\lambda\bar{p} = \mu\lambda\bar{p}$. □

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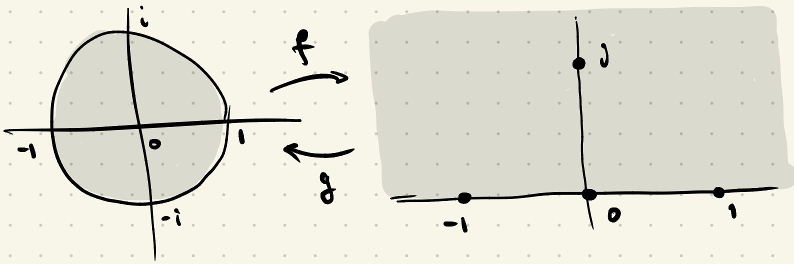
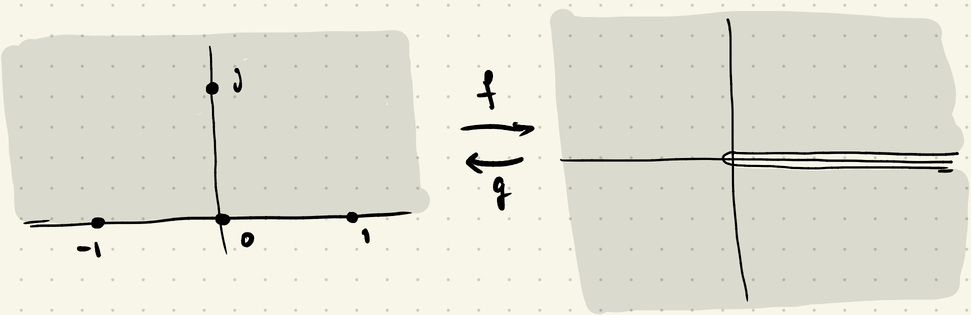
(6) MANY EXAMPLES WRITE $U \cong V$ FOR CONFORMAL EQUIV.

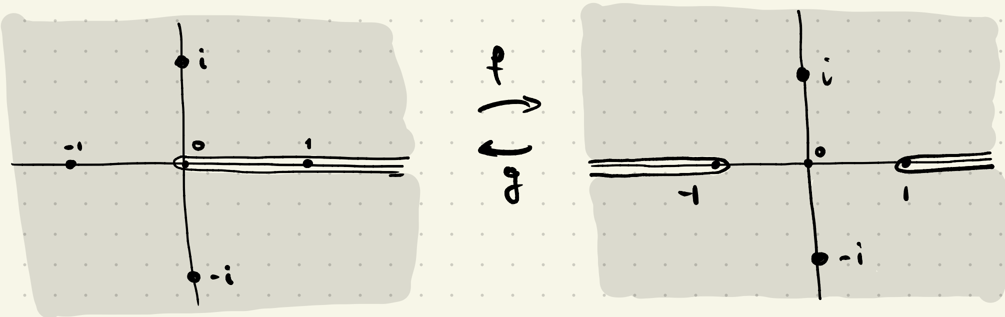
(A) $\mathbb{D} \cong \mathbb{H}$ VIA $f(z) = \dfrac{z+i}{iz+1}$, $g(w) = \dfrac{w-i}{-iw+1}$



(B) $\mathbb{H} \cong \mathbb{C} - \mathbb{R}_{\geq 0}$ VIA $f(z) = z^2$, $g(w) = \sqrt{w}$

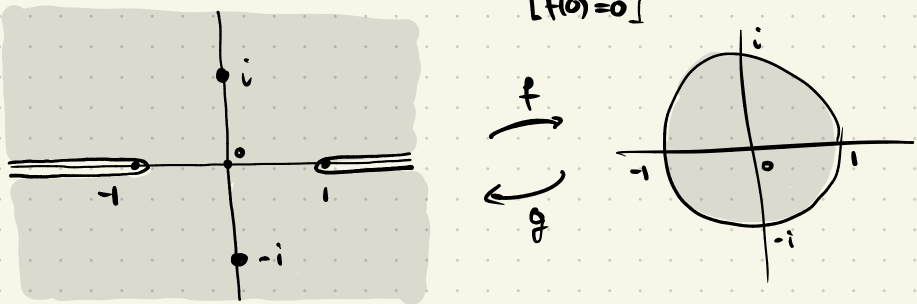


(c) $\mathbb{C} - \mathbb{R}_{\geq 0} \cong \mathbb{C} - (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1})$ VIA $f(z) = \frac{z+1}{-z+1}$, $g(w) = \frac{w-1}{w+1}$



(D) $\mathbb{C} - (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1}) \cong \mathbb{D}$ VIA $f(z) = \frac{1-\sqrt{1-w^2}}{w}$, $g(w) = \frac{2}{w+1/w}$

$[f(0)=0]$



HARDER:

THEOREM SUPPOSE $0<a<b<\infty$, $0<p<q<\infty$

THEN: $A(0;a,b) \cong A(0;p,q)$ IFF $b/a = p/q$.