① SCHWARZ'S LEMMA: NEEDED FOR CLASSIFICATION of f IN $\mathrm{AUT}(\mathbb{D})$.

LEMMA: SUPPOSE $f : \mathbb{D} \to \mathbb{D}$ HOLOMORPHIC AND $f(0)=0$.

THEN (i) $|f(z)| < |z|$ FOR ALL $z \in \mathbb{D}$.

AND (ii) $|f'(0)| \leq 1$

MOREOVER: IF (i) $|f(z)| = |z|$ FOR ANY $z \in \mathbb{D}^\times$

OR (ii) $|f'(0)| = 1$

THEN f IS A ROTATION.

PROOF: IF f CONSTANT THEN $f=0$ AND DONE. SO SUPPOSE f NON CONSTANT. SINCE $f(0)=0$ $\mathrm{ORD}(f,0) \geq 1$. FACTOR TO GET $g : \mathbb{D} \to \mathbb{C}$ SO THAT $f(z) = z \cdot g(z)$. SO $g(z) = \begin{cases} f(z)/z, & z \neq 0 \\ f'(0), & z=0 \end{cases}$

FIX ANY $z_0 \in \mathbb{D}^\times$. FIX ANY $r>0$ SO THAT $|z_0| < r < 1$.

WE APPLY THE MAX. MOD. PRINCIP. TO g ON $\overline{B(0;r)}$

WE FIND $|g(z_0)| \leq |g(z_r)|$

SO $$\frac{|f(z_0)|}{|z_0|} = |g(z_0)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}$$

SO $|f(z_0)| \leq \frac{|z_0|}{r}$ AS r TENDS TO 1 WE OBTAIN $|f(z_0)| \leq |z_0|$.

NOTE $|f'(0)| = \lim_{z \to 0} \left|\frac{f(z)}{z}\right| = \lim_{z \to 0} |g(z)|$ IS THUS AT MOST ONE.

NOW SUPPOSE $|f(z_0)| = |z_0|$ OR $|f'(0)| = 1$. THEN MAX MOD of $|g|$ REALISED IN $\mathbb{D}$. SO g CONSTANT. SO $\frac{f(z)}{z} = a \in \mathbb{C}$.

SO $|a| = 1$ $\square$.

⑤ $\underline{\mathrm{AUT}(\mathbb{D}) = \mathrm{AUT}_{\mathbb{D}}}\ \underline{(\hat{\mathbb{C}}) =}\ \rho\underline{(\mathrm{PSU}(1,1))}$

LEMMA: SUPPOSE $f(z) = \dfrac{az+b}{\bar{b}z+\bar{a}}$ WHERE $|a|^2 - |b|^2 = 1$. THEN $f \in \mathrm{AUT}(\mathbb{D})$. FURTHERMORE EVERY $h \in \mathrm{AUT}(\mathbb{D})$ IS of THIS FORM.

PROOF: $f \in \mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}})$ BY EXERCISES.

$\mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}}) \subset \mathrm{AUT}(\mathbb{D})$ BY RESTRICTION.

SUPPOSE $h \in \mathrm{AUT}(\mathbb{D})$. SUPPOSE $h(0) = 0$. LET $g = h^{-1}$ [ALSO IN $\mathrm{AUT}(\mathbb{D})$.]. SO $h \circ g = \mathrm{Id}_{\mathbb{D}}$ AND SO

$$h'(g(0)) \cdot g'(0) = 1$$

NOTE $g(0) = 0$. SO $|h'(0)| \cdot |g'(0)| = 1$. BY SCHWARZ $|h'(0)| \leq 1$, $|g'(0)| \leq 1$. SO $|h'(0)| = |g'(0)| = 1$. SO BOTH ROTATIONS. SUPPOSE $h(z) = \lambda^2 z$. SO $h(z) = \dfrac{\lambda z + 0}{0z + 1/\lambda}$

SINCE $1/\lambda = \bar{\lambda}$, h HAS DESIRED FORM.

SUPPOSE INSTEAD $h(0) = p$. DEFINE $g: \mathbb{D} \to \mathbb{D}$ BY

$$g(z) = \frac{z - p}{-\bar{p}z + 1}$$

NOTE $1 - |p|^2 > 0$. TAKE $\mu = \dfrac{1}{\sqrt{1-|p|^2}}$ [REAL]

SO $g(z) = \dfrac{\mu z - \mu p}{-\mu\bar{p}z + \mu}$ LIES IN $\rho(\mathrm{PSU}(1,1))$.

NOW: $g \circ h : \mathbb{D} \to \mathbb{D}$ LIES IN $\mathrm{AUT}(\mathbb{D})$ AND FIXES ZERO.

SO $f = g \circ h$ IS A ROTATION, SAY $f(z) = \dfrac{\lambda z + 0}{0 + 1/\lambda} = \lambda^2 z$.

SO $h(z) = (g^{-1} \circ f)(z) = g^{-1}(\lambda^2 z) = \dfrac{\mu\lambda^2 z + \mu p}{\mu\bar{p}\,\lambda^2 z + \mu}$

$$= \frac{\mu\lambda z + \mu\bar{\lambda} p}{\mu\bar{p}\lambda z + \mu\bar{\lambda}} \quad [\bar{\lambda} = 1/\lambda]$$

NOTE $\bar{\mu} = \mu \in \mathbb{R}$. SO $\overline{\mu\lambda} = \mu\bar{\lambda}$, $\overline{\mu\bar{\lambda} p} = \bar{\mu}\lambda\bar{p} = \mu\lambda\bar{p}$. $\square$

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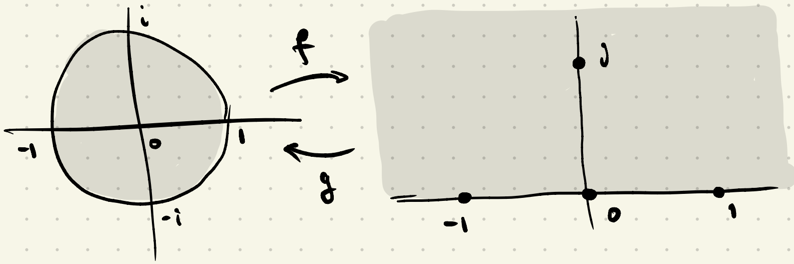
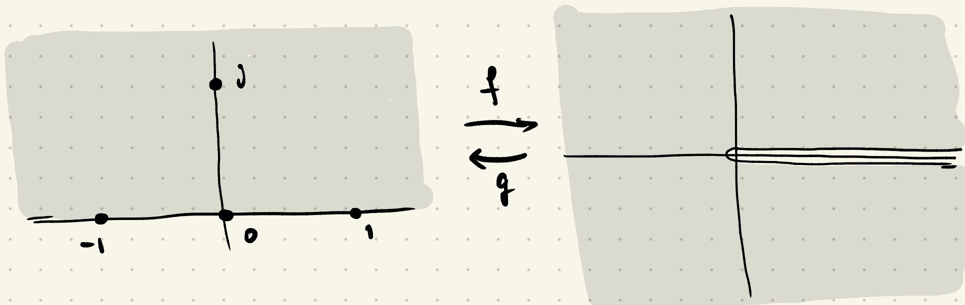
(6) MANY EXAMPLES WRITE $U \cong V$ FOR CONFORMAL EQUIV.

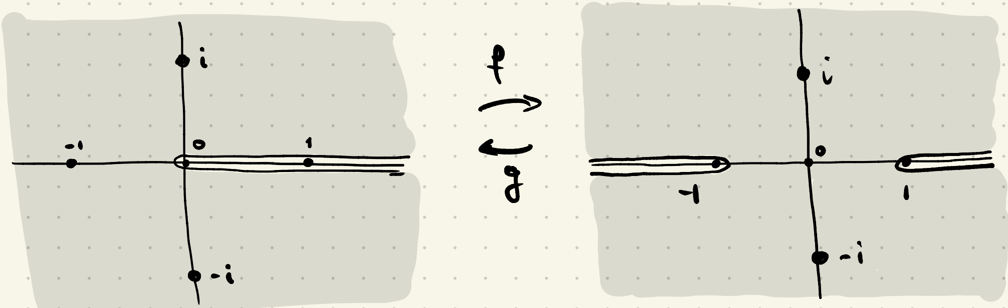
(A) $\mathbb{D} \cong \mathbb{H}$ VIA $f(z) = \frac{z+i}{iz+1}$, $g(w) = \frac{w-i}{-iw+1}$



(B) $\mathbb{H} \cong \mathbb{C} - \mathbb{R}_{\geq 0}$ VIA $f(z) = z^2$, $g(w) = \sqrt{w}$

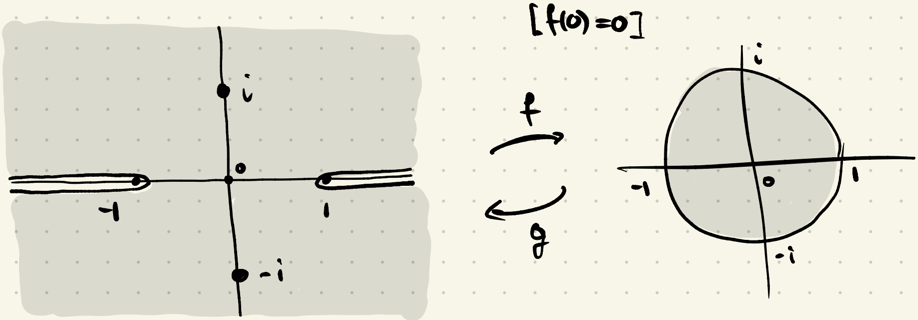


(C) $\mathbb{C} - \mathbb{R}_{\geq 0} \cong \mathbb{C} - (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1})$ VIA $f(z) = \frac{z+1}{-z+1}$, $g(w) = \frac{w-1}{w+1}$



(D) $\mathbb{C} - (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1}) \cong \mathbb{D}$ VIA $f(z) = \dfrac{1-\sqrt{1-w^2}}{w}$, $g(w) = \dfrac{2}{w + 1/w}$

$[f(0) = 0]$



[HARDER] SUPPOSE $a,b,c,d \in \mathbb{R}$, $0 < a < b$, $0 < c < d$.
THEN $A(0;a,b) \cong A(0;c,d)$ IFF $b/a = d/c$.
[IF $a=0$ THEN NEED $c=0$. IF $a=0$, $b=\infty$ NEED $c=0, d=\infty$]

[MUCH EASIER]: $\mathrm{AUT}(\mathbb{C} - \{0,1\})$ IS FINITE.

__HINT__: IN FACT, $\mathrm{AUT}(\mathbb{C} - \{0,1\}) \cong \mathrm{AUT}_{\{\infty,0,1\}}(\hat{\mathbb{C}}) \cong \mathrm{SYM}(3)$.

__STANDARD EXERCISES__: "SUPPOSE U,V DOMAINS. DETERMINE IF U BIHOLOMORPHIC TO V. DETERMINE $\mathrm{AUT}(U)$, $\mathrm{AUT}(V)$."

__POINT__: $\mathrm{HOLO}(U,V) = \{f : U \to V \mid f \text{ HOLOMORPHIC}\}$ ADMITS A $\left\{\begin{array}{l}\text{LEFT}\\ \text{RIGHT}\end{array}\right\}$ ACTION BY $\left\{\begin{array}{l}\mathrm{AUT}(V)\\ \mathrm{AUT}(U)\end{array}\right\}$ VIA $\begin{array}{l} f \cdot h = f \circ h \\ g \cdot f = g \circ f. \end{array}$

__DIAGRAM__:

$$\begin{tikzcd} U \arrow[loop above, "g"] \arrow[r, "f"] & V \arrow[loop above, "h"] \end{tikzcd}$$

PRE- POST-COMPOSE