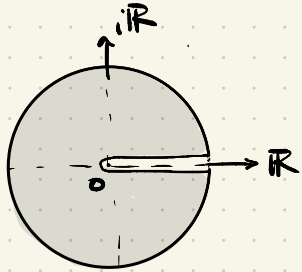
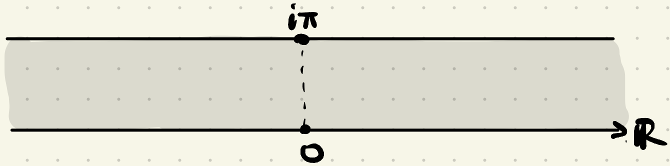
① EVERTHING IS SECRETLY $\mathbb{D}$

MORE EXAMPLES:

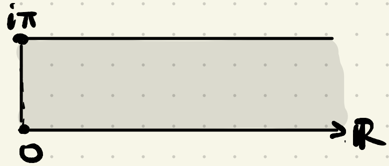
(A) THE SLIT DISC : $\mathbb{D} - [0,1)$



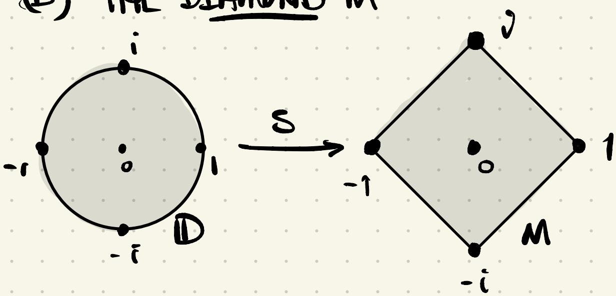
(B) THE STRIP $S = \{ z \in \mathbb{C} \mid 0 < \text{IMAG}(z) < \pi \}$



(C) THE HALF STRIP $H = \{ z \in \mathbb{C} \mid 0 < \text{IMAG}(z) < \pi,\ 0 < \text{REAL}(z) \}$



(D) THE DIAMOND M



SET $\dfrac{\varpi}{2} = \displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} \approx 1.31102\ldots$

DEFINE $S : \mathbb{D} \to M$ BY:

$$S(w) = \frac{2}{\varpi} \int_0^w \frac{dz}{\sqrt{1-z^4}} \qquad \square$$

② THE RIEMANN MAPPING THEOREM

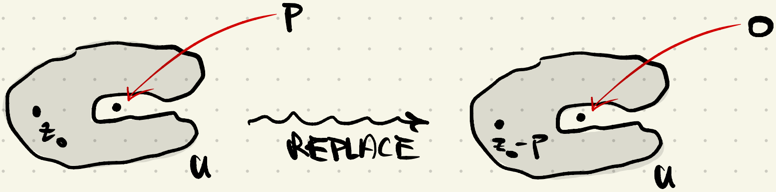
THM: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $U \neq \mathbb{C}$ AND U HAS TRIVIAL FIRST HOMOLOGY. THEN THERE IS A CONFORMAL EQUIV. $f : U \to \mathbb{D}$.

PROOF: WE SLOWLY IMPROVE U.

STEP 1: SUPPOSE $p \notin U$. DEFINE $f : \mathbb{C} \to \mathbb{C}$ BY $f(z) = z - p$

REPLACE U BY $f(U)$. SO NOW $0 \notin U$.

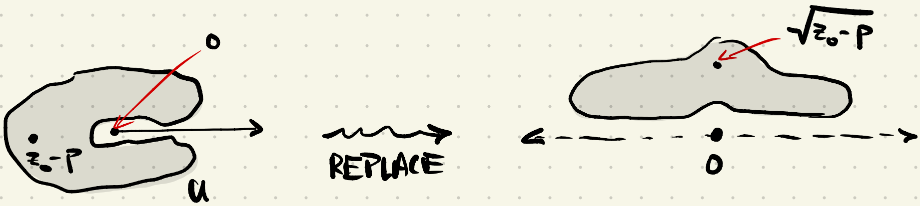
FIGURE



STEP 2: LET $f : U \to \mathbb{C}$ BE A BRANCH of $\sqrt{z}$.

[EXISTS BECAUSE $H_1(U) \cong 0$ AND $0 \notin U$].

REPLACE U BY $f(U)$. SO NOW $w \in U$ IFF $-w \notin U$.

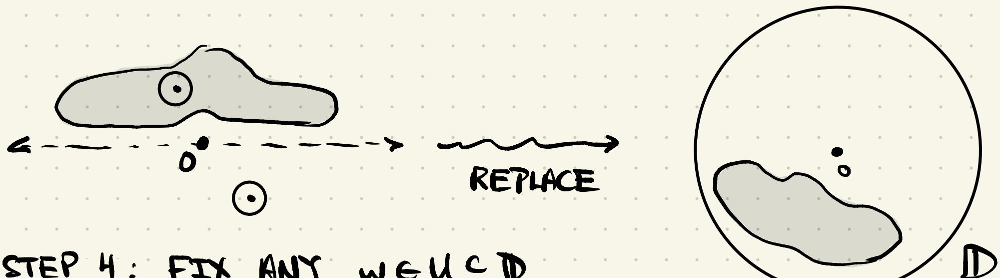


[AND $0 \notin U$.]

STEP 3: FIX $w_0 \in U$. SO HAVE $\epsilon > 0$ WITH $\overline{B(w_0; \epsilon)} \subset U$.

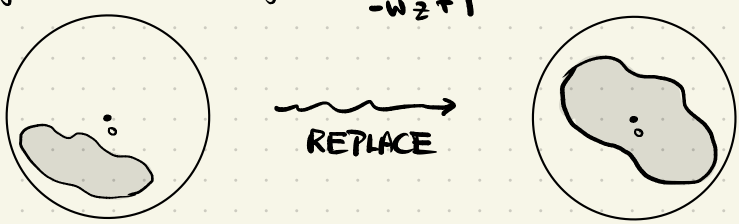
SO $\overline{B(-w_0; \epsilon)} \cap U = \phi$. DEFINE $f : U \to \mathbb{C}$ BY

$f(z) = \dfrac{\epsilon}{z + w_0}$. REPLACE U BY $f(U) \subset \mathbb{D}$.



STEP 4: FIX ANY $w \in U \subset \mathbb{D}$.

DEFINE $f : \mathbb{D} \to \mathbb{D}$ BY $f(z) = \dfrac{z - w}{-\bar{w} z + 1}$. REPLACE U BY $f(U)$.



SUMMARY: WE NOW HAVE $U \subset \mathbb{D}$ WITH $0 \in U$.

DEFINE

$$\mathcal{H} = \left\{ f : U \longrightarrow \mathbb{D} \;\middle|\; \begin{array}{c} f \text{ HOLOMORPHIC,} \\ \text{INJECTIVE } f(0)=0 \end{array} \right\}$$

NOTE $Id_U \in \mathcal{H}$ SO $\mathcal{H} \neq \emptyset$.

DEFINE $D = D(\mathcal{H}) = \sup \{ |f'(0)| \mid f \in \mathcal{H} \}$.

NOTE $Id_U \in \mathcal{H}$ SO $\underline{D(\mathcal{H}) \geq 1}$

FIX $\varepsilon > 0$ SO THAT $\overline{B(0;\varepsilon)} \subset U$. SO $C(0;\varepsilon) \subset U$.

APPLY CAUCHY ESTIMATE: FOR ANY $f \in \mathcal{H}$ HAVE

$$f'(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^2} \, dz$$

SO $|f'(0)| \leq 1/\varepsilon$. SO $D(\mathcal{H}) \leq 1/\varepsilon$.

LEMMA (*): SUPPOSE $U \subset \mathbb{D}$, $0 \in U$, $U \neq \mathbb{D}$. THEN THERE IS $H : U \longrightarrow \mathbb{D}$, HOLOMORPHIC, INJECTIVE, SO THAT

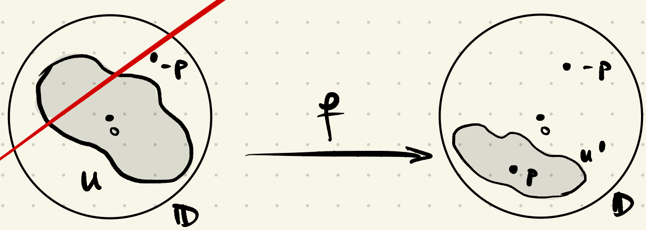
(*) $H(0) = 0$

(*) $|H'(0)| > 1$.

PROOF: SUPPOSE $-p \in \mathbb{D} - U$.

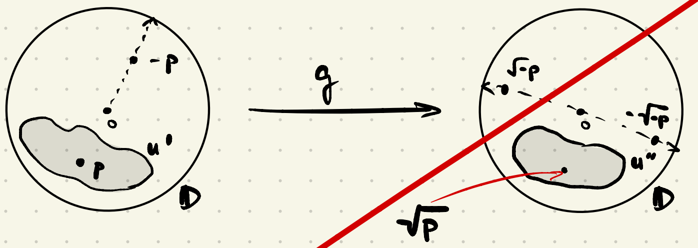
DEFINE $f : \mathbb{D} \to \mathbb{D}$ BY $f(z) = \dfrac{z+p}{\bar{p}z+1}$

DEFINE $U' = f(U)$. SO $p \in U'$, $0 \notin U'$, $U' \subset \mathbb{D}$



DEFINE $g: U' \to \mathbb{D}$ BY $g(z) = \sqrt{z}$ [BRANCH of SQ. ROOT]

DEFINE $U'' = g(U')$. SO $0 \notin U''$, $U'' \subset \mathbb{D}$.



DEFINE $h: \mathbb{D} \to \mathbb{D}$ BY $h(z) = \dfrac{z - \sqrt{p}}{-\sqrt{\bar{p}}\, z + 1}$

DEFINE $U''' = h(U'')$. SO $0 \in U'''$, $U''' \subset \mathbb{D}$.

RECAP:

$$h(z) = \frac{z - \sqrt{p}}{-\sqrt{\bar{p}}\, z + 1} \qquad g(z) = \sqrt{z} \qquad f(z) = \frac{z + p}{\bar{p} z + 1}$$

SO:

$$h'(z) = \frac{1 - |p|}{(-\sqrt{\bar{p}}\, z + 1)^2} \qquad g'(z) = \frac{1}{2}\frac{1}{\sqrt{z}} \qquad f'(z) = \frac{1 - |p|^2}{(\bar{p} z + 1)^2}$$

RECALL:

$$0 \xrightarrow{f} p \xrightarrow{g} \sqrt{p} \xrightarrow{h} 0 \quad \Big\} \text{ DEFINE } H = h \circ g \circ f.$$

$$h'((g \circ f)(0)) = \frac{1}{1 - |p|} \qquad g'(f(0)) = \frac{1}{2}\frac{1}{\sqrt{p}} \qquad f'(0) = 1 - |p|^2$$

SO: $H'(0) = h'((g \circ f)(0)) \cdot g'(f(0)) \cdot f'(0)$

$$= \frac{1}{\cancel{1 - |p|}} \cdot \frac{1}{2}\frac{1}{\sqrt{p}} \cdot (1 + |p|)\cancel{(1 - |p|)} = \frac{1}{2}\,\frac{1 + |p|}{\sqrt{p}}.$$

SO: $|H'(0)| = \dfrac{1/\sqrt{|p|} + \sqrt{|p|}}{2} > 1.$ $\square$

LEMMA

NOW FIX $(f_n) \subset \mathcal{H}$ SO THAT $\lim_{n\to\infty} |f_n'(0)| = D(\mathcal{H})$

LEMMA [MONTEL'S THM] PASSING TO A SUBSEQ AND REINDEXING WE HAVE $f_n \longrightarrow f$ [UNIF CONV. ON COMPACT SETS, $f: \mathbb{D} \to \overline{\mathbb{D}}$ HOLOMORP.]

PROOF: FIX $B(z_0;\epsilon)$ WITH $\overline{B(z_0;\epsilon)} \subset U$. PASS TO SUBSEQ AND REINDEX SO $f_n(z_0)$ CONVERGES. NOTE $|f_n'(z_0)| \leq \frac{1}{\epsilon}$ BY CAUCHY ESTIMATE. SO PASS TO SUBSEQ SO THAT $f_n'(z_0)$ CONVERGES. ... IN GENERAL $|f_n^{(k)}(z_0)| \leq 1/\epsilon^k$. SO APPLY DIAGONAL ARGUMENT TO ARRANGE ALL DERIVATIVES CONVERGING. SO GET $f: B(z_0;\epsilon) \longrightarrow \mathbb{C}$ AS LIMIT OF SUBSEQ OF (f_n). NOW COVER U BY COUNTABLY MANY DISCS OF FORM $B(z_k;\epsilon_k)$ AND AGAIN USE DIAGONAL ARGUMENT. $\square$

NOTE $|f'(0)| = D(\mathcal{H})$ BY CONSTRUCTION.
SO f NON-CONSTANT. SO OPEN MAPPING THM IMPLIES $f(U)$ OPEN. SO $f(U) \subset \mathbb{D}$ [$f(U) \cap \partial\mathbb{D} = \emptyset$].
ALSO, $f(0) = 0$.

LEMMA [HURWITZ] f IS INJECTIVE.
PROOF: VERY SIMILAR TO PROOF OF OPEN MAPPING THEOREM. READ NOTES. $\square$

SO: $f \in \mathcal{H}$. THAT IS, f REALISES THE MAXIMUM POSSIBLE (MODULUS OF) DERIVATIVE AT ZERO.

FINALLY: $f(U) = \mathbb{D}$. FOR, IF NOT, $f(U) \neq \mathbb{D}$ SO APPLY LEMMA * TO FIND H WITH $H: f(U) \longrightarrow \mathbb{D}$ AND $H \circ f \in \mathcal{H}$,

$$|(H \circ f)'(0)| = |H'(f(0)) \cdot f'(0)|$$
$$= |H'(0)| \cdot |f'(0)|$$
$$= |H'(0)| \cdot D(\mathcal{H})$$
$$> D(\mathcal{H}),$$

A CONTRADICTION.

$\square$.