LECTURE 29 2025-12-12 MA3B8 SAUL SCHLEIMER

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① PROVING RIEMANN MAPPING THEOREM

THM: SUPPOSE $U \subset \mathbb{C}$ DOMAIN, $H_1(U) \cong 0$, $U \neq \mathbb{C}$.
THEN U CONF EQUIV TO $\mathbb{D}$.

PROOF SO FAR: WE'VE REDUCED TO THE CASE WHERE $U \subset \mathbb{D}$, $0 \in U$.

DEFINE:
$$\mathcal{H} = \{ f : U \to \mathbb{D} \mid f \text{ HOLO, INJ, } f(0)=0 \}.$$

THIS IS THE "FAMILY" of "HOLOMORPHIC REIMBEDDINGS" of U INTO $\mathbb{D}$.

DEFINE: $D(\mathcal{H}) = \sup \{ |f'(0)| : f \in \mathcal{H} \}$.

NOTE $\mathrm{Id}_U \in \mathcal{H}$, SO $D(\mathcal{H}) \geq 1$.

FIX $\varepsilon > 0$ SO THAT $\overline{B(0;\varepsilon)} \subset U$. BY CAUCHY ESTIMATE

FOR ANY $f \in \mathcal{H}$, FOR $f(z) = \sum a_k z^k$ IN $B(0;\varepsilon)$
WE HAVE $|a_k| \leq 1/\varepsilon^k$. SO $|a_1| \leq 1/\varepsilon$. SO $D(\mathcal{H}) \leq 1/\varepsilon$.

FIX $(f_n)_n \subset \mathcal{H}$ SO THAT $\lim\limits_{n \to \infty} |f_n'(0)| = D(\mathcal{H})$.

MONTEL'S THM: THERE IS A SUBSEQ $(f_{n_k})_k$ WHICH CONVERGES (UNIF ON COMP SUBSETS) TO SOME HOLO $f_\infty : U \to \mathbb{D}$.

PROOF: APPLY CAUCHY EST, SO $|f_n^{(k)}(0)| \leq k!/\varepsilon^k < \infty$.
PASS TO SUBSEQ (AND THEN TAKE DIAGONAL) TO ENSURE THAT $(f_n^{(k)}(0))_n$ CONVERGES FOR ALL $k \in \mathbb{N}$.
LET $f_\infty : B(0;\varepsilon) \longrightarrow \mathbb{C}$ BE THE LIMITING FUNCTION.

NOTE f_∞ HOLOMORPHIC, NON-CONSTANT. NOW COVER U BY COUNTABLY MANY CLOSED DISCS, PASS TO SUBSEQUENCES, TAKE THE DIAGONAL, AND SO EXTEND f_∞ TO $f_\infty : U \to \mathbb{C}$

NOTE IMAGE$(f) \subset \overline{\mathbb{D}}$. SINCE f_∞ IS NON CONSTANT, $f_\infty(U)$ OPEN [BY OPEN MAPPING THM]. SO $f_\infty(U) \subset \mathbb{D}$. □

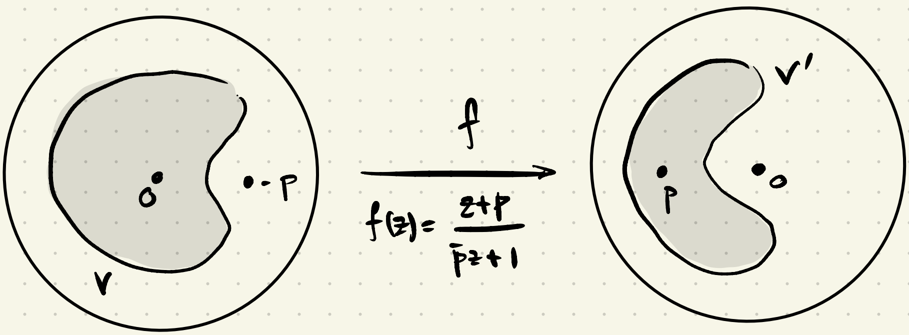
<u>HURWITZ THM</u>: SUPPOSE $f_n \longrightarrow f_\infty$ (AS ABOVE) WITH ALL f_n HOLOMORPHIC, INJECTIVE. THEN f_∞ IS HOLO, INJECTIVE.

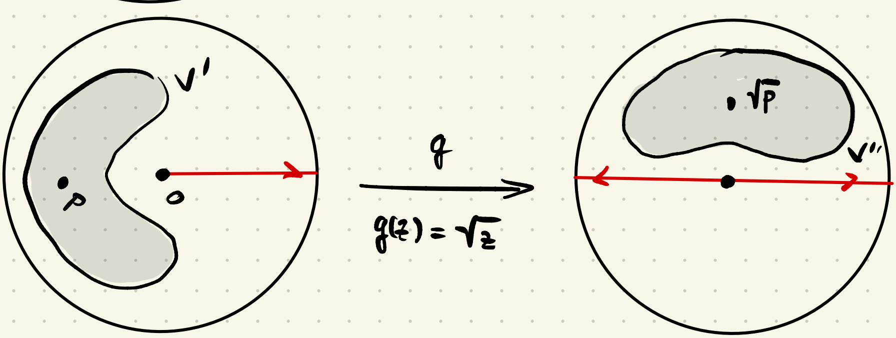
<u>PROOF</u>: SAME AS PROOF OF OPEN MAPPING THM, PLUS A BIT. □

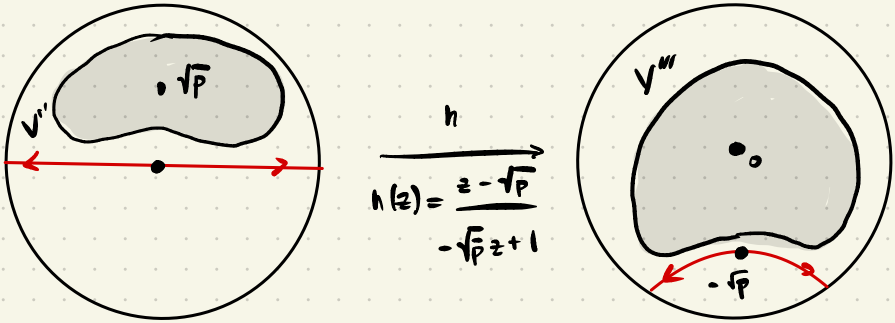
<u>SO</u>: $f_n \longrightarrow f_\infty$ IMPLIES f_∞ HOLO, INJ. SO $f_\infty \in \mathcal{H}$.
SO $|f_\infty'(0)| = D(\mathcal{H})$ MAXIMAL POSSIBLE. IF $f_\infty(U) = \mathbb{D}$ WE ARE DONE. SO: SUPPOSE, FOR A CONTRADICTION THAT $V = f_\infty(U) \neq \mathbb{D}$.

<u>A CONSTRUCTION</u>: [EXPANSION LEMMA]

HERE ARE THE PICTURES.







SO:

$$f(z) = \frac{z+p}{\bar{p}z+1} \qquad g(z) = \sqrt{z} \qquad h(z) = \frac{z-\sqrt{p}}{-\sqrt{\bar{p}}\,z+1}$$

$$f(0) = p \qquad g(f(0)) = \sqrt{p} \qquad h(g(f(0))) = 0$$

$$f'(z) = \frac{1-|p|^2}{(\bar{p}z+1)^2} \qquad g'(z) = \frac{1}{2\sqrt{z}} \qquad h'(z) = \frac{1-|p|}{(-\sqrt{\bar{p}}\,z+1)^2}$$

$$f'(0) = 1-|p|^2 \qquad g'(f(0)) = \frac{1}{2\sqrt{p}} \qquad h'(g(f(0))) = \frac{1-|p|}{(1-|p|)^2} = \frac{1}{1-|p|}$$

SO: SET $H = h\circ g\circ f : V \to \mathbb{D}$, $H(0)=0$.

$$H'(0) = h'(g(f(0)))\cdot g'(f(0))\cdot f'(0) = \frac{1+|p|}{2\sqrt{p}}$$

so $$|H'(0)| = \frac{|p|^{1/2}+|p|^{-1/2}}{2} > 1.$$

FINALLY: $H\circ f_\infty \in \mathcal{H}$ AND

$$|(H \circ f_\infty)'(0)| = |H'(f_\infty(0))| \cdot |f_\infty'(0)|$$
$$= |H'(0)| \cdot D(\mathcal{H}) > D(\mathcal{H}).$$

THIS IS A CONTRADICTION. SO $f_\infty(U) \neq \mathbb{D}$ IS FALSE.
SO $f_\infty(U) = \mathbb{D}$, AS DESIRED. $\blacksquare$.