MA3B8: COMPLEX ANALYSIS

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0. Introduction

...the shortest and best way between two truths of the real domain often passes through the imaginary one.

– Jacques Hadamard
The psychology of invention in the mathematical field

These notes set out the "material of record" for the 2025-2026 module MA3B8 (complex analysis) at Warwick. We give solutions to all exercises (except one) in Appendix A (starting on page 76). We also give a short index of notation (on page 96) and a list of references (on page 97).

If you find any errors in the notes, or any possible improvements, please inform me directly (at s.schleimer@warwick.ac.uk) or use the anonymous form:

`https://forms.gle/AnXkWf5DM8epZgF8A`

Acknowledgements. I thank the many teachers, colleagues, and students without whom these notes would not exist. I particularly owe a debt to Donald Sarason and Curt McMullen; their courses at Berkeley were where I first learned this remarkable subject. Sarason's book [9] gives a wonderful exposition of the topic for an advanced undergraduate. I have also spent many a happy hour reading Ahlfors' classic text [1]. Those two books are the foundation underlying the house inside of which these notes were written.

1. Rectangular and polar coordinates

1.1. **Notation.** We use the following notations.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.
- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are, respectively, the sets of integers, rationals, and reals.
- $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ is the set of complex numbers.

We assume the usual algebraic properties of all of these. If a and b are real numbers with $a \leq b$ then we use $[a, b]$ to denote the closed interval with a and b as endpoints.

We assume the usual topological and geometric properties of the plane $\mathbb{R}^2$; this includes the idea of homeomorphism of subsets. When convenient we will identify $\mathbb{R}$ with either the x–axis in $\mathbb{R}^2$ or the real axis in $\mathbb{C}$.

We also assume the usual analytic properties of real-valued functions of several variables, giving references where needed. We assume that $\exp, \cos, \sin\colon \mathbb{R} \to \mathbb{R}$ are given by their power series; we further assume that their usual properties (including derivatives and identities) are available.

1.2. **Rectangular coordinates.** Suppose that

$$z = x + iy$$

is complex. Then we define

$$\textsc{Real}(z) = x \quad \text{and} \quad \textsc{Imag}(z) = y$$

to be the *real* and *imaginary* parts of z. We also define

$$\bar{z} = x - iy$$

to be the *complex conjugate* of z. We define

$$|z| = \sqrt{x^2 + y^2}$$

to be the *magnitude* of z. Note that $z\bar{z} = |z|^2$. Also, $|z| = 0$ if and only if $z = 0$.

Suppose also that $r > 0$ is a positive real number. We define

$$B(z; r) = \{w \in \mathbb{C} : |w - z| < r\}$$

and

$$\overline{B(z; r)} = \{w \in \mathbb{C} : |w - z| \leq r\}$$

Thus $B(z; r)$ is the open, and $\overline{B(z; r)}$ is the closed, *ball* of radius r about z. (These are also called the open and closed *discs* of radius r about z.) We call $B^\times(z, r) = B(z; r) - \{z\}$ the *punctured* open ball of radius r about z. Similarly we call $\mathbb{C}^\times = \mathbb{C} - \{0\}$ the *punctured* plane.

We define

$$C(z; r) = \{w \in \mathbb{C} : |w - z| = r\}$$

Thus $C(z;r)$ is the *circle* of radius r about z. (We also call $C(z;r)$ the *boundary* of $\overline{B(z;r)}$.) As a bit of notation we take $\mathbb{D} = B(0;1)$ and $S^1 = C(0;1)$: the *unit disc* and the *unit circle*. We also use $\mathbb{D}^\times = \mathbb{D} - \{0\}$ for the *punctured unit disc*. We also define $\mathbb{H} = \{z \in \mathbb{C} \mid \text{IMAG}(z) > 0\}$: the *upper-half plane*.

Suppose that Z is a subset of the plane. We define

$$N(Z;r) = \{w \in \mathbb{C} : |w - z| < r \text{ for some } z \in Z\}$$

and

$$\overline{N(Z;r)} = \{w \in \mathbb{C} : |w - z| \leq r \text{ for some } z \in Z\}$$

to be, respectively, the *open* and *closed* r*–neighbourhoods* of Z. Note that $N(Z;r)$ is the union of the open r–balls about points z of Z. The same holds for $\overline{N(Z;r)}$ with closed replacing open.

1.3. **Polar coordinates.** Suppose that $z = x+iy \neq 0$. Thus $x^2+y^2 > 0$. So we may form

$$C = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad S = \frac{y}{\sqrt{x^2+y^2}}$$

Note that $C^2 + S^2 = 1$. We say that $\theta \in \mathbb{R}$ is an *argument* of z if

$$\cos(\theta) = C \quad \text{and} \quad \sin(\theta) = S$$

We define $\text{ARG}(z)$ to be the set of arguments of z. From the definitions we deduce that, for any $\theta \in \text{ARG}(z)$, we have the following *polar form*:

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

As a useful bit of notation we write $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. The addition law implies that $e^{i\theta+i\eta} = e^{i\theta}e^{i\eta}$.

Exercise 1.4. Suppose that $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Prove that $\text{ARG}(z)$ is non-empty. Prove that, for any argument θ, we have $\text{ARG}(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$. ◇

Thus the argument is a *multi-valued function*: a one-to-many relation. This subtlety lies at the heart of the theory of Riemann surfaces.

Exercise 1.5. Suppose that $z, w \in \mathbb{C}^\times$. Prove that

$$|zw| = |z||w| \qquad \text{and} \qquad \text{ARG}(zw) = \text{ARG}(z) + \text{ARG}(w)$$ ◇

Exercise 1.6. Suppose that $z, w \in \mathbb{C}^\times$. Suppose that θ and η are arguments of z and w, respectively. Prove the cosine law

$$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta)$$

directly from the definitions and the addition law. ◇

With arguments and the cosine law in hand, we can identify the geometry of $\mathbb{C}$ with that of $\mathbb{R}^2$. In particular, addition of complex numbers is "the same as" addition of vectors, multiplication by a complex number is "the same as" scaling-plus-rotation, and arguments (correctly chosen) are "the same as" the angles of various triangles.

Definition 1.7. Suppose that a and b are complex numbers, with $a \neq 0$. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto az + b$. Then we call f an *(orientation preserving) similarity*. If $|a| = 1$ then f is an *(orientation preserving) congruence*. If $a = 1$ then f is a *translation*. If $b = 0$ then f is a *homothety* about the origin. If a is real and positive, and $b = 0$, then f is a *dilation* about the origin. If $|a| = 1$ and $b = 0$ then the function is a *rotation* about the origin. ◇

Exercise 1.8. Show that $\textsc{Sim}(\mathbb{C})$, the set of similarities of $\mathbb{C}$ together with the operation of function composition, is a group. Show that each subclass (congruence, translation, homothety, dilation, and rotation) gives a subgroup. ◇

Suppose that U and V are subsets of the plane. We say that U and V are *similar* if there is a similarity sending U to V. We make the same definition for the various subclasses in Definition 1.7.

Exercise 1.9. Suppose that $z, w \in \mathbb{C}$. Suppose that $z \neq 0$ and that $|z - w| < |z|$. Prove that $w \neq 0$. Prove, furthermore, that for any $\theta \in \textsc{Arg}(z)$ there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi$. (In fact the bound can be improved to $\pi/2$.) ◇

We now prove that "the" argument is "continuous" on the punctured plane $\mathbb{C}^\times$.

Lemma 1.10. *Suppose that $z \in \mathbb{C}^\times$. Suppose that $\epsilon > 0$. Then there is a $\delta > 0$ with the following property.*

Suppose that $w \in \mathbb{C}$ has $|z - w| < \delta$. Then $w \neq 0$ and, for every $\theta \in \textsc{Arg}(z)$, there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \epsilon$.

Proof. We suppose (as we may) that $\epsilon < \pi/2$. We take $\delta = \sin(\epsilon)|z|$. Thus δ is positive and less than $|z|$. So $|z-w| < \delta < |z|$. By Exercise 1.9 we have $w \neq 0$ and we have a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi$.

Since $|z - w|^2 < \delta^2 = |z|^2 \sin^2(\epsilon)$ the cosine law gives us

$$|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta) < |z|^2 \sin^2(\epsilon)$$

Moving terms and simplifying gives

$$|z|^2 \cos^2(\epsilon) + |w|^2 < 2|z||w|\cos(\theta - \eta)$$

Subtracting from both sides gives

$$|z|^2 \cos^2(\epsilon) - 2|z||w|\cos(\epsilon) + |w|^2 < 2|z||w|\cos(\theta - \eta) - 2|z||w|\cos(\epsilon)$$

Thus

$$(|z|\cos(\epsilon) - |w|)^2 < 2|z||w|(\cos(\theta - \eta) - \cos(\epsilon))$$

The left-hand side is non-negative. Replacing it by zero and simplifying gives

$$\cos(\epsilon) < \cos(\theta - \eta)$$

Since cosine is even, and decreasing on $[0, \pi]$, we deduce that $|\theta - \eta| < \epsilon$, as desired. □

2. Contour integrals

I keep a close watch on this heart of mine
I keep my eyes wide open all the time
I keep the ends out for the tie that binds
Because you're mine, I walk the line

– Jonny Cash
I walk the line

2.1. Domains.

Definition 2.2. Suppose that $U \subset \mathbb{C}$ is non-empty, open, and connected. Then we call U a *domain*. ◇

Domains can be quite wild; see Figure 2.3.

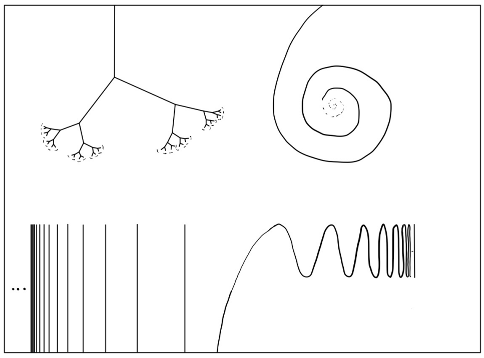

Figure 2.3. A wild domain. Here U is the open rectangle minus four closed sets: a *dendrite* (upper left), a *spiral* (upper right), a *comb* (lower left), and a *topologist's sine curve* (lower right).

Nonetheless, domains have several nice properties.

Exercise 2.4. Domains are path-connected. ◇

(More is true; see Lemma 7.3.)

Exercise 2.5. Suppose that $P \subset U$ is compact and non-empty. Prove that there is a constant $\epsilon > 0$ so that the open neighbourhood $N(P; \epsilon)$ is contained in U. ◇

With domains in hand, we can propose a solution to the problem of multi-valued functions.

Definition 2.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that F is a multi-valued function on U. A *branch* of F is

- a domain $V \subset U$ and
- a continuous function $f \colon V \to \mathbb{C}$

so that, for all $z \in V$, we have $f(z) \in F(z)$. ◇

For example, Arg has a branch $\arg \colon \mathbb{H} \to \mathbb{R}$ obtained by always picking $\arg(z) \in (0, \pi)$. The continuity of arg follows, say, from the differentiability of the tangent function and the inverse function theorem. We return to this in Section 20.1.

2.7. **Contours.**

Definition 2.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $[a, b]$ is a closed interval in $\mathbb{R}$. A *contour in* U is a continuous function $\gamma\colon [a, b] \to U$ with:

- a finite sequence
$$(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b) \subset [a, b]$$
so that, for all k, the restriction $\gamma|[t_k, t_{k+1}]$ is continuously differentiable with non-vanishing derivative.

We call the t_i *breakpoints* and their images $\gamma(t_i)$ *corners* of γ. At a corner $t \in [a, b]$ we may use $\gamma'_-(t)$ and $\gamma'_+(t)$ for the left and right derivatives. ◇

Given our identification of $\mathbb{R}^2$ and $\mathbb{C}$ we may write $\gamma'(t)$ as a complex number.

As an example of a contour in $\mathbb{C}$ we have $\gamma\colon [0, 2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = e^{i\theta}$. The derivative is $\gamma'(\theta) = ie^{i\theta}$.

2.9. **Integrals.**

Definition 2.10. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a, b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We define the *contour integral* of f along γ as follows.

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

If γ has corners then we break the right-hand side into the corresponding finite sum of integrals. ◇

Example 2.11. Suppose $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma\colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = e^{i\theta}$. Suppose that $f\colon U \to \mathbb{C}$ is given by $f(z) = 1/z$. Then we have:

$$\int_\gamma f\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$ ◇

Example 2.12. Suppose that w is a point of $\mathbb{C}$. Suppose $U = \mathbb{C} - \{w\}$ is a punctured plane. Suppose that $r > 0$ is real. Suppose that $\gamma\colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = re^{i\theta} + w$. So $\gamma'(\theta) = ire^{i\theta}$. Suppose that $f\colon U \to \mathbb{C}$ is given by $f(z) = 1/(z - w)$. Then we have:

$$\int_\gamma f\,dz = \int_\gamma \frac{dz}{z - w} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

So this particular contour integral is insensitive to translation of the singularity and to rescaling of the circle. ◇

The contour integral is linear with respect to the integrand, as follows.

Lemma 2.13. *Suppose that U, γ, and f are as in Definition 2.10. Suppose that $g\colon U \to \mathbb{C}$ is a continuous function. Suppose that $p, q \in \mathbb{C}$ are complex numbers. Then we have:*

$$\int_\gamma (p \cdot f + q \cdot g)\, dz = p \int_\gamma f\, dz + q \int_\gamma f\, dz \qquad \square$$

The contour integral is also linear with respect to the domain of integration, as follows.

Lemma 2.14. *Suppose that U, γ, and f are as in Definition 2.10. Suppose that c is a point of $[a, b]$. Define $\gamma_a = \gamma|[a, c]$ and $\gamma_b = \gamma|[c, b]$. Then we have:*

$$\int_\gamma f\, dz = \int_{\gamma_a} f\, dz + \int_{\gamma_b} f\, dz \qquad \square$$

The contour integral is almost independent of parametrisation, as follows.

Lemma 2.15. *Suppose that U, γ, and f are as in Definition 2.10. Suppose that $\delta\colon [c, d] \to [a, b]$ is piecewise C^1 with nonvanishing derivative. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have:*

$$\int_{\gamma\circ\delta} f\, dz = \int_\gamma f\, dz$$

If, on the other hand, $\delta(c) = b$ and $\delta(d) = a$, then

$$\int_{\gamma\circ\delta} f\, dz = -\int_\gamma f\, dz$$

In the first case we say that γ and $\gamma \circ \delta$ have the *same orientation*. In the second case we say that they have *opposite orientations*.

Proof of Lemma 2.15. We compute as follows.

$$\begin{aligned}\int_{\gamma\circ\delta} f\, dz &= \int_c^d f(\gamma(\delta(u)))\, (\gamma \circ \delta)'(u)\, du \\ &= \int_c^d f(\gamma(\delta(u)))\, \gamma'(\delta(u))\, \delta'(u)\, du \\ &= \int_a^b f(\gamma(t))\, \gamma'(t)\, dt = \int_\gamma f\, dz\end{aligned}$$

If δ reverses the endpoints then we instead obtain the following.

$$= \int_b^a f(\gamma(t))\, \gamma'(t)\, dt = -\int_\gamma f\, dz \qquad \square$$

Exercise 2.16. A *Laurent polynomial over* $\mathbb{C}$ is a finite sum $f = \sum_{k=m}^n a_k z^k$ where $m \leq n$ are integers and a_k is a complex number for all k. Suppose that U and γ are as in Example 2.11. Prove the following.

$$\int_\gamma f\, dz = 2\pi i a_{-1} \qquad \diamond$$

2.17. **Arc-length.** We define the *element of arc-length* by:

$$ds^2 = dx^2 + dy^2 \quad \text{and so} \quad ds = \sqrt{dx^2 + dy^2}$$

This is used as follows.

Definition 2.18. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a, b] \to U$ is a contour. We define the *arc-length* of γ as follows.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\, dt \qquad \diamond$$

Example 2.19. The upper half of the unit circle is parametrised by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, \pi]$. Since this has unit speed, we have the following.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_0^\pi |\gamma'(\theta)|\, d\theta = \int_0^\pi d\theta = \pi$$

The upper half of the circle is also parametrised by $\delta(t) = (t, \sqrt{1 - t^2})$ for $t \in [-1, 1]$. Computing, we have the following.

$$\begin{aligned} L(\delta) &= \int_\delta \sqrt{dx^2 + dy^2} = \int_{-1}^1 |\delta'(t)|\, dt = \int_{-1}^1 \sqrt{1 + \frac{t^2}{1 - t^2}}\, dt \\ &= \int_{-1}^1 \frac{1}{\sqrt{1 - t^2}}\, dt = \int_{-\pi/2}^{\pi/2} d\theta = \pi \qquad \diamond \end{aligned}$$

As the above example suggests, the arc-length integral is independent of parametrisation.

Lemma 2.20. *Suppose that U and γ are as in Definition 2.18. Suppose that $\delta\colon [c, d] \to [a, b]$ is piecewise C^1 and is bijective. Then we have $L(\gamma \circ \delta) = L(\gamma)$.*

Exercise 2.21. Prove Lemma 2.20. $\diamond$

We now have a very easy but extremely useful bound: the *ML–inequality.*

Lemma 2.22. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a, b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is continuous. We take*

$$M = \max\{|f(\gamma(t))| : t \in [a, b]\}$$

Then we have:

$$\left| \int_\gamma f\, dz \right| \le M \cdot L(\gamma)$$

Exercise 2.23. Prove Lemma 2.22. $\diamond$

3. Variation of argument

Man: *Ah. I'd like to have an argument, please.*
Receptionist: *Certainly sir. Have you been here before?*
Man: *No, I haven't, this is my first time.*
Receptionist: *I see. Well, do you want to have just one argument, or were you thinking of taking a course?*

– Monty Python
Argument Clinic

We have two ways to understand the variation of argument: analytical and topological.

3.1. **The analytical viewpoint.** Define the *element of argument* to be

$$d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

This can be derived by differentiating $\tan(\theta) = y/x$. This is used as follows.

Definition 3.2. Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour in the punctured plane. We define the *variation of argument* of γ about zero as follows.

$$V(\gamma) = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_a^b \frac{\gamma_x(t)\gamma_y'(t) - \gamma_y(t)\gamma_x'(t)}{|\gamma(t)|^2}\,dt \qquad \diamond$$

For example, the variation of argument about zero for the upper half-circle (oriented anticlockwise) is π; the variation for the whole circle is 2π.

Lemma 3.3. *Suppose that γ is a contour in the punctured plane $\mathbb{C}^\times$. Suppose that $\delta\colon [c,d] \to [a,b]$ is piecewise C^1. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have $V(\gamma \circ \delta) = V(\gamma)$. If, instead, $\delta(c) = b$ and $\delta(d) = a$ then $V(\gamma \circ \delta) = -V(\gamma)$.* □

The proof is similar to that of Lemma 2.15 and we omit it.

Proposition 3.4. *Suppose that γ is a contour in the punctured plane $\mathbb{C}^\times$. Suppose that $\theta_a \in \mathrm{Arg}(\gamma(a))$ is an argument. Then there exists a unique continuous $\theta\colon [a,b] \to \mathbb{R}$ so that*

- *$\theta(a) = \theta_a$ and*
- *$\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ for all $t \in [a,b]$.*

Proof. Induction on the number of corners reduces the general case to the case where there are none. So we assume that γ is C^1. For s in $[a,b]$ we define $\gamma_s = \gamma|[a,s]$. Next we define

$$\theta(s) = \theta_a + V(\gamma_s) = \theta_a + \int_{\gamma_s} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

We deduce from the definitions that $V(\gamma) = \theta(b) - \theta(a)$. We also deduce, from the fundamental theorem of calculus, that θ is differentiable (as a real function of s). Thus θ is continous.

We define $r(s) = |\gamma(s)|$. We now claim that $\gamma(s) = r(s)e^{i\theta(s)}$ for all $s \in [a, b]$. To prove this, consider

$$h(s) = \frac{r(s)e^{i\theta(s)}}{\gamma(s)}$$

Differentiating h we find that $h'(s) = 0$ for all s.

Exercise 3.5. Check this. $\diamond$

Also $h(a) = 1$; thus $h(s) = 1$ for all s. This implies that $\gamma(s) = r(s)e^{i\theta(s)}$ for all s.

Suppose that $\eta\colon [a, b] \to U$ is another such function. That is, $\eta(a) = \theta_a$ and $\gamma(t) = r(t)e^{i\eta(t)}$ for all $t \in [a, b]$. We deduce that $\theta(t) - \eta(t)$ is continuous and also lies in $2\pi\mathbb{Z}$ for all t. Thus $\theta(t) - \eta(t)$ is constant. However $\theta(a) - \eta(a) = 0$, so $\eta = \gamma$, as desired. $\square$

A contour $\gamma\colon [a, b] \to U$ is *closed* if $\gamma(b) = \gamma(a)$. (This is a confusing but extremely common abuse of terminology.)

Lemma 3.6. *Suppose that γ is a closed contour in the punctured plane $\mathbb{C}^\times$. Then we have $V(\gamma) \in 2\pi\mathbb{Z}$.*

Proof. Recall that $\gamma(b) = \gamma(a)$. Thus $e^{i\theta(b)} = e^{i\theta(a)}$. The addition law, and periodicity, tells us that $\theta(b) - \theta(a)$ lies in $2\pi\mathbb{Z}$, as desired. $\square$

Definition 3.7. Suppose that γ is a closed contour in the punctured plane $\mathbb{C}^\times$. We define

$$\textsc{Wind}(\gamma) = V(\gamma)/2\pi$$

to be the *winding number* of γ about zero. $\diamond$

Remark 3.8. Suppose that $z_0 \in \mathbb{C}$. Suppose that γ is a contour in the punctured plane $\mathbb{C} - \{z_0\}$. Then we define $V(\gamma, z_0)$, the variation of argument of γ about z_0; it is variation of the contour $t \mapsto \gamma(t) - z_0$ about zero.

Suppose that γ is closed. Then we define $\textsc{Wind}(\gamma, z_0)$, the winding number of γ about z_0; it is the winding number of $t \mapsto \gamma(t) - z_0$ about zero. $\diamond$

3.9. **The topological viewpoint.**

Second proof of Proposition 3.4. Recall that the image of γ avoids the origin. For every $t \in [a, b]$ we choose some r_t so that the ball $B(\gamma(t); r_t)$ misses the origin. Let I_t be the component of $\gamma^{-1}(B(\gamma(t); r_t))$ that contains t. So $\mathcal{U} = \{I_t\}$ is an open cover of $[a, b]$ by intervals. Since $[a, b]$ is compact, we may choose a finite subcover $\mathcal{V} \subset \mathcal{U}$. Let $T \subset [a, b]$ be the finite set of times t so that I_t lies in $\mathcal{V}$.

We now define a continuous function $\theta\colon [a,b] \to \mathbb{R}$ by induction. We first define $\theta(a) = \theta_a$. Suppose now that we have defined the function $\theta|[a,c]$ on the interval $[a,c] \subset [a,b]$. Suppose additionally that $\theta|[a,c]$ is continuous and satisfies $\theta(s) \in \mathrm{A{\small RG}}(\gamma(s))$ for all $s \in [a,c]$.

If $c = b$ then take $\theta = \theta|[a,b]$ and the definition is complete. Suppose instead that $c < b$. Now, c lies in (the interior of) I_t for some $t \in T$. Note that $\gamma(I_t)$ lies in $B(\gamma(t); r_t)$. We apply Exercise 1.9; for all s in (the closure of) I_t there is a unique argument $\theta(s) \in \mathrm{A{\small RG}}(\gamma(s))$ so that $|\theta(s) - \theta(c)| < \pi$. This extends $\theta|[a,c]$ to $[a,c']$, the (closure of) $[a,c] \cup I_t$. By Lemma 1.10 the extension $\theta|[a,c']$ is continuous. By construction $\theta(s)$ lies in $\mathrm{A{\small RG}}(\gamma(s)$ for all $s \in [a,c']$. Finally, since T is finite this process terminates, giving the desired $\theta|[a,b]$.

The argument for the uniqueness of θ (subject to the condition that $\theta(a) = \theta_a$) is identical to one given above. ☐

3.10. **Element of the logarithm.** We end this section three more examples. The *element of radius* is as follows.

$$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

This can be derived by differentiating $r^2 = x^2 + y^2$. Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour in the punctured plane. Then integrating the element of radius over γ gives the difference $|\gamma(b)| - |\gamma(a)|$.

The *element of logarithmic radius* is:

$$\frac{dr}{r} = \frac{x\,dx + y\,dy}{x^2 + y^2}$$

This can be derived by differentiating $\log(r)$. Integrating this over γ, a contour in $\mathbb{C}^\times$, gives the ratio $|\gamma(b)|/|\gamma(a)|$. As in the proof of Exercise 3.5 we have

$$\frac{dr}{r} + i\,d\theta = \frac{dz}{z}$$

This is called the *element of the logarithm.*

4. Holomorphic functions

4.1. **Limits in $\mathbb{C}$.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $g\colon U - \{z_0\} \to \mathbb{C}$ is a function. Then we write

$$\lim_{z \to z_0} g(z) = c$$

if, for all real $\epsilon > 0$ there is some real $\delta > 0$ with the following property: if $0 < |z - z_0| < \delta$ then $|g(z) - c| < \epsilon$.

We reiterate - it is not enough to consider sequences tending to z_0 along a particular line, or finite collection of lines. Instead, we must allow z to tend to z_0 inside of a shrinking sequence of (punctured) balls.

We now have the central definition in complex analysis.

Definition 4.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. We say that f is *holomorphic* if, for all $z_0 \in U$, the limit
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists. ◇

We denote the limit at z_0 by $f'(z_0)$. Since f is holomorphic in all of U, we obtain a function $f'\colon U \to \mathbb{C}$. We call f' the *complex derivative* of f. Note that we do *not* assume that f' is continuous.

Example 4.3. The usual rules of limits apply. Thus the sum and product of holomorphic functions are again holomorphic, and their derivatives behave as expected. As a consequence, any polynomial $P \in \mathbb{C}[z]$ is holomorphic. ◇

Example 4.4. As a non-example, consider $f(z) = \bar{z}$. The derivative at zero, taken along the real axis, is 1. The derivative at zero, taken along the imaginary axis, is -1. Thus $f(z) = \bar{z}$ is not holomorphic. (It is sometimes called "anti-holomorphic".) ◇

Example 4.5. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic, and g is never zero. Then the ratio f/g is also holomorphic. Its derivative satisfies the usual quotient rule. As a consequence, rational functions (ratios of polynomials) are holomorphic away from the zeros of their denominators. ◇

As a special case of this, we have an exceedingly important example: the function $f(z) = 1/z$ is holomorphic in the punctured plane $\mathbb{C}^\times$.

Example 4.6. Suppose that $U, V, W \subset \mathbb{C}$ are domains. Suppose that $f\colon U \to V$ and $g\colon V \to W$ are holomorphic. Then $g \circ f$ is also homomorphic, and its derivative $(g' \circ f) \cdot f'$ is given by the chain rule. ◇

Exercise 4.7. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f is continuous. ◇

The following observation is "trivial" but surprisingly useful.

Lemma 4.8. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Fix $z_0 \in U$. Define $\rho\colon U \to \mathbb{C}$ by*
$$\rho(z) = \begin{cases} 0 & z = z_0 \\ \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & z \neq z_0 \end{cases}$$
Then ρ is continuous.

Exercise 4.9. Prove Lemma 4.8. ◇

Finally, holomorphic functions are differentiable when thought of as real-valued functions of two variables.

Lemma 4.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then, for any $w \in U$, the total derivative $D_w f$ exists and equals the matrix*

$$D_w f = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

where $r = |f'(w)|$ and θ is an argument of $f'(w)$.

Proof. The partial derivative $\partial/\partial x$ is obtained by requiring z to tend to w while remaining inside of a horizontal line. Since f is holomorphic, the limit exists, and agrees with $f'(w)$. The same holds for $\partial/\partial y$, replacing horizontal by vertical. □

We deduce that, when non-zero, the total derivative Df is a homothety (that is, scaling and rotating). Thus f, away from the zeros of its derivative, is *conformal*: preserves signed angles.

4.11. **Power series.** We quickly review the theory of power series.

Definition 4.12. Suppose that z_0 is a complex number. Suppose that $(a_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers. Then we write

$$\sum_{k\in\mathbb{N}} a_k(z - z_0)^k$$

for the *power series* with *centre* z_0 and *coefficients* (a_k). We write $f_n(z) = \sum_{k=0}^{n} a_k(z - z_0)^k$ for the n^{th} *partial sum.* We also define

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k}$$

We take $R = 0$ or $R = \infty$ when the limit superior is infinite or zero, respectively. We call R the *radius of convergence* of the given power series. ◇

Ahlfors [1, Theorem 2, page 38] attributes the following result to Abel.

Theorem 4.13. *Suppose that z_0, $(a_n)_n$, f_n, and R are as in Definition 4.12. Suppose that $R > 0$. Suppose that $0 < R' < R$. Then:*

- *The power series converges absolutely for all z in $B(z_0; R)$.*

Let f be the resulting function on $B(z_0; R)$.

- *The partial sums f_n converge uniformly to f on $\overline{B(z_0; R')}$.*
- *The function f is holomorphic in $B(z_0; R)$.*
- *Its derivative $f'(z)$ has the power series*

$$\sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

which again has radius of convergence R.

- *The function f has complex derivatives of all orders at z_0; its n^{th} derivative is $f^{(n)}(z_0) = n! \cdot a_n$.* □

With f as in the theorem, we say that f has a *series expansion* in $B(z_0; R)$. Note that the function f also has a series expansion in $B(z_0; R')$ for all $R' < R$. The algebra of functions and power series "agree", as follows.

Theorem 4.14. *Suppose that z_0 and R are as in Definition 4.12. Suppose that $R > 0$. Suppose that $f, g\colon B(z_0; R) \to \mathbb{C}$ are functions. Suppose that f and g have series expansions with coefficients $A = (a_n)_n$ and $B = (b_n)_n$, respectively. Then:*

- *f and g are equal if and only if A and B are equal.*
- *The function $f + g$ has a series expansion with coefficients $(a_n + b_n)_n$.*
- *The function $f \cdot g$ has a series expansion with coefficients $C = (c_n)_n$ where*

$$c_n = \sum_{k+\ell=n} a_k b_\ell$$

- *Suppose that $f = (z - z_0)^k \cdot g$. Then we have:*

$$a_n = \begin{cases} 0, & \text{for } n < k \\ b_{n-k} & \text{for } n \geq k \end{cases}$$

- *Suppose that $f(z_0) \neq 0$. Then the function $h = 1/f$ has a series expansion with positive radius of convergence and with coefficients $c = (c_n)_n$ satisfying*

$$a_0 c_0 = 1 \quad \text{and, for } n > 0 \quad \sum_{k+\ell=n} a_k c_\ell = 0$$

Exercise 4.15. Prove Theorem 4.14. ◇

Exercise 4.16. Give a power series about zero for the function $f(z) = 1/(1 + z^2)$. Determine its radius of convergence. ◇

Exercise 4.17. Give a power series about zero for the function $f(z) = 1/(1 + z + z^2)$. Determine its radius of convergence. ◇

We define the complex versions of the exponential and the trigonometric functions as follows.

Definition 4.18. We define

$$\begin{aligned}
\textsc{Exp}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\textsc{Cos}(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\
\textsc{Sin}(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}
\end{aligned}$$ ◇

These agree with their real siblings when restricted to the real axis in $\mathbb{C}$. The family resemblances extend much further.

Lemma 4.19. *The complex exponential and trigonometric functions have the following properties. (Here z and w range over $\mathbb{C}$ unless otherwise stated.)*

- $\textsc{Cos}(0) = 1$ *and* $\textsc{Sin}(0) = 0$.
- $\textsc{Cos}(2\pi) = 1$ *and* $\textsc{Sin}(2\pi) = 0$.
- $\textsc{Cos}'(z) = -\textsc{Sin}(z)$ *and* $\textsc{Sin}'(z) = \textsc{Cos}(z)$.
- $\textsc{Cos}^2(z) + \textsc{Sin}^2(z) = 1$.
- $\textsc{Exp}(iz) = \textsc{Cos}(z) + i\,\textsc{Sin}(z)$.
- $\textsc{Exp}(0) = 1$ *and* $\textsc{Exp}(2\pi i) = 1$.
- $\textsc{Exp}'(z) = \textsc{Exp}(z)$.
- $\textsc{Exp}(z + w) = \textsc{Exp}(z)\,\textsc{Exp}(w)$.
- *The addition laws:*

$$\begin{aligned}
\textsc{Cos}(z+w) &= \textsc{Cos}(z)\,\textsc{Cos}(w) - \textsc{Sin}(z)\,\textsc{Sin}(w) \\
\textsc{Sin}(z+w) &= \textsc{Cos}(z)\,\textsc{Sin}(w) + \textsc{Sin}(z)\,\textsc{Cos}(w)
\end{aligned}$$

- $\textsc{Cos}(z + 2\pi) = \textsc{Cos}(z)$ *and* $\textsc{Sin}(z + 2\pi) = \textsc{Sin}(z)$.
- *Furthermore, all periods of* $\textsc{Cos}$ *and* $\textsc{Sin}$ *are multiples of* 2π.
- $\textsc{Exp}(z + 2\pi i) = \textsc{Exp}(z)$.
- *Furthermore, all periods of* $\textsc{Exp}$ *are multiples of* $2\pi i$.
- $\textsc{Exp}(z) \neq 0$.
- *For all* $w \in \mathbb{C}^\times$ *there is some* $z \in \mathbb{C}$ *so that* $\textsc{Exp}(z) = w$.
- $\textsc{Exp}(-z) = 1/\textsc{Exp}(z)$.
- $\textsc{Exp}$ *is a surjective homomorphism from the group* $(\mathbb{C}, +)$ *to the group* $(\mathbb{C}^\times, \cdot)$, *with kernel* $(2\pi i\mathbb{Z}, +)$. □

Exercise 4.20. Prove that the zeros of the complex cosine and sine are all real (and so agree with the zeros of the real cosine and sine, respectively). ◇

Exercise 4.21. Prove that the complex exponential takes "rectangular coordinates" (in its domain) to "polar coordinates" (in its image). That is, it takes horizontal line segments to radial ones and vertical line segments to arcs of circles (centred on the origin).

Suppose that $0 < a < b$ are real numbers. Sketch the closed rectangle R with vertices at $\log(a)$, $\log(b)$, $\log(b) + 2\pi i$, and $\log(a) + 2\pi i$. Sketch the image of R and its boundary under the complex exponential. $\diamond$

In view of the above, for $w \in \mathbb{C}^\times$ we define

$$\textsc{Log}(w) = \{z \in \mathbb{C} \mid \textsc{Exp}(z) = w\}$$

"The" complex logarithm is a "multi-valued function", like "the" argument, and for the same reason.

Lemma 4.22. *Suppose that $w \in \mathbb{C}^\times$. Then we have:*

$$\textsc{Log}(w) = \log(|w|) + i\,\textsc{Arg}(w)$$ $\square$

5. Triangles

5.1. **Segments.** Suppose that p and q are complex numbers. We use the notation

$$[p, q] = \{(1 - t)p + tq \mid t \in [0, 1]\}$$

to denote the *line segment* between p and q. The segment is a set. However, we also treat it as a contour using the linear parametrisation $t \mapsto (1 - t)p + tq = p + t(q - p)$. Note that $[p, q]$ and $[q, p]$ are identical as sets but have opposite orientations as contours.

Example 5.2. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} dz = \int_0^1 (q - p)\, dt = q - p$$

More generally, if $\gamma\colon [a, b] \to U$ is any contour, then $\int_\gamma dz = \gamma(b) - \gamma(a)$. $\diamond$

Example 5.3. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} z\, dz = \int_0^1 (p+t(q-p))(q-p)\, dt = p(q-p)+\frac{1}{2}(q-p)^2 = \frac{1}{2}(q^2-p^2)$$

More generally, if $\gamma\colon [a, b] \to U$ is any contour, then $\int_\gamma z\, dz = \frac{1}{2}(\gamma(b)^2 - \gamma(a)^2)$. $\diamond$

5.4. **Convexity.** Suppose that K is a subset of the complex plane. We say that K is *convex* if, for every $p, q \in K$, the line segment $[p, q]$ lies in K. As a first example, the plane itself is convex.

Exercise 5.5. Prove that $B(z_0; r)$ and $\overline{B(z_0; r)}$ are convex. $\diamond$

Suppose that S is a set of complex numbers. We define the *convex hull* of S to be the intersection of all convex sets containing S.

Exercise 5.6. Prove that the convex hull of S is convex. (This justifies the name. We also deduce that the convex hull of S is the smallest convex set containing S.) $\diamond$

Suppose that p, q, and r are complex numbers. The *triangle* $T = T(p, q, r)$ is the convex hull of $\{p, q, r\}$. By Exercise 5.6 the line segments $[p, q]$, $[q, r]$, and $[r, p]$ lie in T; these are the *sides* of T. We define the *boundary* ∂T to be the union of $[p, q]$, $[q, r]$, and $[r, p]$.

Suppose that $T = T(p, q, r)$ is a triangle. We will, very often, need to realise ∂T as a contour; as such ∂T it traverses the segments $[p, q]$, $[q, r]$, and $[r, p]$ in that order and with those orientations. If p, q, and r lie in a line in $\mathbb{C}$ then we call T a *degenerate triangle.*

Definition 5.7. Suppose that T is non-degenerate. We say that T is *positively oriented* if the boundary contour ∂T travels around T in anticlockwise fashion. Similarly, T is *negatively oriented* if ∂T travels around T in clockwise fashion. ◇

Lemma 5.8. *Suppose that a, b, p, q, and r are complex numbers $\mathbb{C}$. Take $f(z) = a + bz$; take $T = T(p, q, r)$. Then we have:*

$$\int_{\partial T} f\,dz = 0$$

Proof. Lemma 2.13 implies the following.

$$\int_{\partial T} (a + bz)\,dz = a \int_{\partial T} dz + b \int_{\partial T} z\,dz$$

Lemma 2.14 implies the following.

$$\int_{\partial T} dz = \int_{[p,q]} dz + \int_{[q,r]} dz + \int_{[r,p]} dz$$
$$\int_{\partial T} z\,dz = \int_{[p,q]} z\,dz + \int_{[q,r]} z\,dz + \int_{[r,p]} z\,dz$$

Examples 5.2 and 5.3 give the following.

$$\int_{\partial T} dz = (q - p) + (r - q) + (p - r) = 0$$
$$\int_{\partial T} z\,dz = \frac{1}{2}[(q^2 - p^2) + (r^2 - q^2) + (p^2 - r^2)] = 0$$

Thus the integral of $f(z) = a + bz$ about ∂T is zero, as desired. □

Remark 5.9. Lemma 5.8 generalises in many ways.

As one possibility, suppose that $f \in \mathbb{C}[z]$ is a polynomial. Suppose that $T \subset \mathbb{C}$ is a triangle. Then $\int_{\partial T} f\,dz = 0$.

As another possibility, fix $R > 0$. Suppose that $f \colon B(0; R) \to \mathbb{C}$ is given by a power series. Suppose that T lies in $B(0; R)$. Recall (Theorem 4.13) that the partial sums f_n converge uniformly to f. Uniform convergence allows us to interchange the integral and the limit, giving $\int_{\partial T} f\,dz = 0$. In fact, the integral of f along any closed contour in γ in $B(0; R)$.

We will give somewhat indirect proofs of these statements. However, more direct proofs are possible, using only the techniques developed up to this point. ◇

5.10. **Midpoint subdivision.**

Definition 5.11. Suppose that $T = T(p, q, r)$ is a triangle. Let

$$p' = (q + r)/2, \quad q' = (r + p)/2, \quad r' = (p + q)/2$$

be the midpoints of the sides of T. We define four triangles as follows:

$$\begin{aligned} P &= T(p, r', q') & Q &= T(r', q, p') \\ R &= T(q', p', r) & S &= T(p', q', r') \end{aligned}$$

The collection of triangles $\{P, Q, R, S\}$ give the *midpoint subdivision* of T. ◇

The triangles P, Q, R, and S are all congruent and all have the same orientation. Also they are all similar to T (under scaling by one-half) and all have the same orientation as T. For a labelled sketch, see Figure 6.4. Note that all four smaller triangles have the same orientation (positive, negative, or degenerated) as T.

Lemma 5.12. *Suppose that T is a triangle in $\mathbb{C}$. Suppose that P, Q, R, and S are the triangles of the midpoint subdivision of T. Then we have*

$$\frac{1}{2}L(\partial T) = L(\partial P) = L(\partial Q) = L(\partial R) = L(\partial S)$$
$$\frac{1}{2}D(\partial T) = D(\partial P) = D(\partial Q) = D(\partial R) = D(\partial S)$$

where L is the arc-length of a contour and D is its diameter. □

Remark 5.13. Midpoint subdivision of line segments is defined similarly. Note that, when we perform midpoint subdivision to a triangle, we obtain midpoint subdivision of its three boundary segments. Also the new internal segments have opposite orientations in pairs and so "cancel". Said another way, "midpoint subdivision commutes with taking boundaries". A version of this is proved in Exercise 6.6. ◇

6. Goursat's lemma

Voici une démonstration du théorème de Cauchy, qui me parait un peu plus simple que les démonstrations habituelles. Elle repose uniquement sur la définition de la dérivée et sur cette remarque que les intégrales définies $\int dz$, $\int z\,dz$, prises le long d'un contour fermé quelconque, sont nulles.

– E. Goursat
Démonstration du théorème de Cauchy

6.1. **Integrating to zero around triangles.**

Definition 6.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We say that f *integrates to zero around triangles* if, for every triangle $T \subset U$, we have

$$\int_{\partial T} f\,dz = 0 \qquad \diamond$$

We state and prove a result of Pringsheim, correcting a result of Goursat, improving upon a result of Cauchy. For references, and an overview of the history, see [3].

Lemma 6.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f integrates to zero around triangles in U.*

Proof. In the proof we will use the following notations. Suppose that $\gamma\colon [a,b] \to U$ is a contour. We define $I(\gamma) = \int_\gamma f\,dz$. As in Definition 2.18, we use $L(\gamma)$ for the arc-length of γ. We also use $D(\gamma)$ for the diameter of the image of γ.

By Exercise 4.7 we have that f is continuous. We now fix a triangle $T = T(p,q,r) \subset U$. We must prove that $I(\partial T) = 0$.

Suppose that P, Q, R, and S are the triangles of the midpoint subdivision of T (Definition 5.11). Recall that ∂T and the boundaries of the other triangles are oriented, as shown in Figure 6.4.

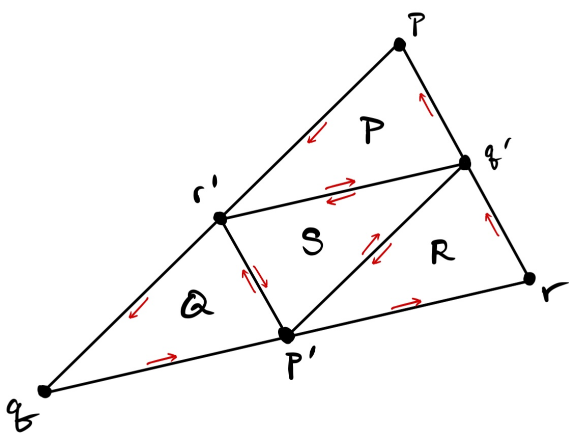

FIGURE 6.4. The midpoint subdivision of T. We label the vertices of all of the triangles. We also decorate the edges of the four smaller triangles with their orientations.

Claim 6.5. $I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$

Proof. We apply Lemma 2.14 and 2.15 and the combinatorics of Figure 6.4. $\square$

Exercise 6.6. Give a more detailed proof of Claim 6.5. $\diamond$

Taking absolute values and applying the triangle inequality, we have

$$|I(\partial T)| \le |I(\partial P)| + |I(\partial Q)| + |I(\partial R)| + |I(\partial S)|$$

Set $T_0 = T$. Pick T_1 to be any one of P, Q, R, or S so that

$$|I(\partial T_1)| = \max\{|I(\partial P)|, |I(\partial Q)|, |I(\partial R)|, |I(\partial S)|\}$$

We deduce that

$$\begin{aligned} |I(\partial T_1)| &\ge \frac{1}{4}|I(\partial T_0)| \\ L(\partial T_1) &= \frac{1}{2}L(\partial T_0) \\ D(\partial T_1) &= \frac{1}{2}D(\partial T_0) \end{aligned}$$

We repeat the above process indefinitely, finding a sequence of triangles $(T_n)_{n=0}^{\infty}$ so that, for all n, we have $T_{n+1} \subset T_n$. Also:

$$\begin{aligned} |I(\partial T_{n+1})| &\ge \frac{1}{4}|I(\partial T_n)| \\ L(\partial T_{n+1}) &= \frac{1}{2}L(\partial T_n) \\ D(\partial T_{n+1}) &= \frac{1}{2}D(\partial T_n) \end{aligned}$$

Since the T_n are nested closed sets, with diameters tending to zero, their intersection contains a single point. Let z_0 that point. Induction also gives us, for all n:

$$\begin{aligned} |I(\partial T_n)| &\ge \frac{1}{4^n}|I(\partial T_0)| \\ L(\partial T_n) &= \frac{1}{2^n}L(\partial T_0) \\ D(\partial T_n) &= \frac{1}{2^n}D(\partial T_0) \end{aligned}$$

Recall that f is the given holomorphic function. We take $\rho\colon U \to \mathbb{C}$ as in Lemma 4.8. Thus ρ is continuous, vanishes at z_0, and satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)$$

Fix any $\epsilon > 0$. We deduce that there is a $\delta > 0$ so that for $w \in \overline{B(z_0;\delta)}$ we have $|\rho(w)| \le \epsilon$. Since the triangles (T_n) nest down to z_0 there is some n so that T_n is contained in $\overline{B(z_0;\delta)}$. We now compute:

$$\begin{aligned}
I(\partial T_n) &= \int_{\partial T_n} f\,dz && \text{definition of } I(\partial T_n)\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z-z_0) + \rho(z)(z-z_0)]\,dz && \text{Lemma 4.8}\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z-z_0)]\,dz + \int_{\partial T_n} \rho(z)(z-z_0)\,dz && \text{Lemma 2.13}\\
&= \int_{\partial T_n} \rho(z)(z-z_0)\,dz && \text{Lemma 5.8}
\end{aligned}$$

Now, the maximum of $|\rho|$ on ∂T_n is at most ϵ. The maximum of $|z-z_0|$ in ∂T_n is at most $D(\partial T_n)$. Thus, by the ML–inequality (2.22) we have:

$$|I(\partial T_n)| \leq \epsilon \cdot D(\partial T_n) \cdot L(\partial T_n) = \frac{\epsilon}{4^n} \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Thus:

$$|I(\partial T_0)| \leq \epsilon \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Since $\epsilon > 0$ was arbitrary, we deduce that $I(\partial T_0) = 0$, as desired. □

Exercise 6.7. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that R is a rectangle in U. Prove that $\int_{\partial R} f\,dz = 0$. (That is, "holomorphic functions integrate to zero around rectangles".) ◇

7. Primitives

7.1. **Piecewise-linear paths.** Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a contour.

We say that γ is *simple* if $\gamma(r) = \gamma(s)$ implies that $x = y$ or $\{x,y\} = \{a,b\}$. (That is, injective contours are simple, and so are closed contours that only fail to be injective at their endpoints.) Recall that we say that γ is *closed* if $\gamma(a) = \gamma(b)$.

We say that γ is *affine* if

$$\gamma(t) = \frac{b-t}{b-a} \cdot \gamma(a) + \frac{t-a}{b-a} \cdot \gamma(b)$$

We say that γ is *piecewise linear* if there are finitely many $(t_k)_{k=0}^N \subset [a,b]$ with $t_0 = a$, with $t_k < t_{k+1}$, with $t_N = b$, and with $\gamma|[t_k, t_{k+1}]$ affine.[1]

A *polygon* $P \subset \mathbb{C}$ is a non-empty compact set, with interior dense in its closure, whose boundary ∂P is a closed and simple piecewise-linear contour. Equipped with these definitions, we can give a motivating special case of Theorem 9.2.

Exercise 7.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $P \subset U$ is a polygon. Prove

[1]Of course this should be called "piecewise affine", but nobody says that.

that $\int_{\partial P} f\,dz = 0$. (That is, "holomorphic functions integrate to zero around polygons".) $\diamondsuit$

We discuss a bit of the topology of the plane.

Lemma 7.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that p and q are distinct points of U. Then there is a simple piecewise-linear contour in U connecting p to q.*

Proof. Let U_p be the subset of points $w \in U$ so that p is connected to w by a piecewise-linear contour in U. Note that balls $B(z_0; \epsilon) \subset \mathbb{C}$ are piecewise-linearly path-connected. We now repeat the solution of Exercise 2.4 line-by-line and deduce that $U_p = U$.

So suppose that $\beta\colon [0,1] \to U$ is a piecewise-linear contour connecting p to q. If β is simple we are done. Suppose not. We now "cut-and-paste": we cut loops out of β and concatenate the pieces which are left.

Here are the details. Set $\beta_0 = \beta$. Suppose that, in our cutting process, we have constructed $\beta_n\colon [0,1] \to U$. By induction we have that β_n is piecewise linear and connects p to q. If β_n is simple we are done.

Suppose not. Thus there are $s, t \in [0,1]$ with the following properties.

- $s < t$,
- $\beta_n(s) = \beta_n(t)$,
- $\beta_n|[0,s]$ is injective, and
- t is the largest in $[0,1]$ with the above properties.

(Note that it is possible that $s = 0$, and it is possible that $t = 1$, but we cannot have both. This is because $p \neq q$.) So $\beta_n|[s,t]$ is a closed contour. Furthermore $\beta_n|[s,t]$ contains at least one segment of β_n. Define β_{n+1} to be the concatenation of $\beta_n|[0,s]$ with $\beta_n|[t,1]$. So β_{n+1} is piecewise linear and connects p to q. Finally, the number of line segments in β_{n+1} is at least one less than the number in β_n.

Since N is finite, the above terminates with the desired simple piecewise-linear contour. $\square$

In fact, more is true; the contour in Lemma 7.3 can be chosen to consist of vertical and horizontal arcs only. We will not need this fact.

7.4. **Basic properties of primitives.** Here is our next definition.

Definition 7.5. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that $F\colon U \to \mathbb{C}$ is holomorphic. Suppose that $F' = f$. Then we say that F is a *primitive* for f in U. $\diamondsuit$

The function F is also called an "antiderivative" for f. Note that if F and G are both primitives for f then $F - G$ is constant.

Lemma 7.6. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then f has a primitive in U if and only if $\int_\gamma f\,dz = 0$ for all closed contours γ in U.*

Proof. Suppose that f has a primitive F. Suppose that $\gamma\colon [a,b] \to U$ is a closed contour. We compute as follows:

$$\int_\gamma f\,dz = \int_\gamma F'\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a)) = 0$$

Suppose instead that f integrates to zero about all closed contours in U. Fix w_0 in U. We now define a function $F\colon U \to \mathbb{C}$: for any $w \in U$ pick a piecewise-linear contour $\alpha\colon [0,1] \to U$ with $\alpha(0) = w_0$ and $\alpha(1) = w$. (Note that α exists by Lemma 7.3.) We define $F(w) = \int_\alpha f\,dz$. Suppose now that β is another such contour. Define γ to be the concatenation of α and the reverse of β:

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2-2t), & 1/2 \le t \le 1 \end{cases}$$

So γ is a closed contour in U. Thus $\int_\gamma f\,dz = 0$. By Lemmas 2.14 and 2.15 we have

$$0 = \int_\gamma f\,dz = \int_\alpha f\,dz - \int_\beta f\,dz \quad \text{and thus} \quad \int_\alpha f\,dz = \int_\beta f\,dz$$

We deduce that F is well-defined.

Suppose that w_1 lies in U. We must prove that $F'(w_1) = f(w_1)$. Fix $\epsilon > 0$. Since f is continuous, and since U is open, we have a δ as follows: if $0 < |w - w_1| < \delta$ then $w \in U$ and $|f(z) - f(w_1)| \le \epsilon$.

Suppose that γ is a piecewise-linear contour from w_0 to w_1. (Again, γ exists by Lemma 7.3.) Let $[w_1, w]$ be the line segment from w_1 to w. Thus $[w_1, w]$ lies in U. Let γ_w be the concatenation of γ and $[w_1, w]$. So

$$F(w_1) = \int_\gamma f\,dz \quad \text{and} \quad F(w) = \int_{\gamma_w} f\,dz$$

Thus

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We subtract $f(w_1)(w - w_1)$ from both sides to find

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} f(z) - f(w_1)\,dz$$

The ML–inequality (2.22) implies that

$$|F(w) - F(w_1) - f(w_1)(w - w_1)| \le \epsilon |w - w_1|$$

Dividing both sides by $|w - w_1|$ gives

$$\left| \frac{F(w) - F(w_1)}{w - w_1} - f(w_1) \right| \le \epsilon$$

Thus $F'(w_1) = f(w_1)$ and we are done. $\square$

Exercise 7.7. Suppose that $f\colon \mathbb{C}^\times \to \mathbb{C}$ is given by $f(z) = 1/z$. Prove that f does not have a primitive in $\mathbb{C}^\times$. ◇

7.8. **Primitives in discs.**

Lemma 7.9. *Suppose that $B = B(w_0; r)$ is a disc. Suppose that $f\colon B \to \mathbb{C}$ is continuous. Suppose that f integrates to zero about all triangles in B. Then f has a primitive in B.*

Note that the disc B cannot be replaced by an arbitrary domain U. As an example, the function $f(z) = 1/z$ integrates to zero about all triangles in $\mathbb{C}^\times$. On the other hand, by Exercise 7.7, we know that f has no primitive in $\mathbb{C}^\times$. On the final hand, Lemma 7.9 implies that f has a primitive in every disc that avoids the origin.

Proof of Lemma 7.9. Let $[w_0, w]$ be the line segment from w_0 to w. We define $F(w) = \int_{[w_0,w]} f\,dz$.

Suppose that w_1 is another point of B. We must prove that $F'(w_1) = f(w_1)$. Since f integrates to zero about triangles, we have:

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We now subtract $f(w_1)(w - w_1)$ from both sides to obtain

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} (f(z) - f(w_1))\,dz$$

The rest of the proof is similar to that of Lemma 7.6. □

8. Homology

8.1. **Simplices and affine maps.** The *standard n–simplex* is

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x_i \geq 0, \sum x_i = 1 \right\}$$

The k^{th} *vertex* v_k of Δ^n is the vector with all coordinates equal to zero, except for the k^{th}, which is equal to one.

Suppose that X is a topological space. A *singular n–simplex in X* is a continuous map $\sigma^n\colon \Delta^n \to X$ from the standard simplex to X. Suppose that W is a real vector space. A singular n–simplex σ^n in W is *affine* if σ^n is the restriction of a linear map.

Lemma 8.2. *Suppose that W is a real vector space. Suppose that (w_i) is an ordered collection of $n+1$ points in W. Then there is a unique affine n–simplex σ^n where $\sigma^n(v_i) = w_i$, for all i.* □

8.3. **Codimension-one faces.** Fix $k \leq n$. We define the list

$$L_k = [v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n]$$

The notation here means that v_k is *omitted* from the list. Thus L_k has n elements, in order.

Suppose that σ^n is a singular n–simplex in X. By the lemma there is a unique singular $(n-1)$–simplex

$$\tau_k^{n-1} \colon \Delta^{n-1} \to \mathbb{R}^{n+1}$$

where τ_k^{n-1} takes the ℓ^{th} vertex of Δ^{n-1} to the ℓ^{th} vertex of L_k. The notation for, and the definition of, the k^{th} $(n-1)$–dimensional *face* of σ^n is:

$$\sigma^n \circ \tau_k^{n-1} = \sigma^n|[v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n]$$

We will always suppress the notation for τ_k^{n-1}. The dimensions relevant to these notes are dimension zero, one, and two.

- A singular zero-simplex σ^0 has no -1–dimensional faces.
- A singular one-simplex σ^1 has two zero-dimensional faces; these are $\sigma^1|[v_1]$ and $\sigma^1|[v_0]$.
- A singular two-simplex σ^2 has three one-dimensional faces; these are $\sigma^2|[v_1, v_2]$, $\sigma^2|[v_0, v_2]$, and $\sigma^2|[v_0, v_1]$.

Exercise 8.4. List the four three-dimensional faces of a singular three-simplex σ^3. ◇

8.5. **Chains and the boundary operator.**

Definition 8.6. Suppose that X is a topological space. Fix n. An *n–chain* in X is a formal, finite sum of singular n–simplices in X with integer coefficients. The collection of all n–chains in X is a commutative group denoted $C_n(X) = C_n(X; \mathbb{Z})$. ◇

In the latter notation the "$\mathbb{Z}$" denotes the coefficient ring for our finite formal sums. Our coefficients will always be integers; thus we will suppress the coefficient ring from our notation.

Definition 8.7. Suppose that X is a topological space. Fix n. Suppose that σ^n is a singular n–chain in X. We define the *homological boundary* of σ^n to be

$$\partial_n \sigma^n = \sum_k (-1)^k \sigma^n|[v_0, v_1, \ldots, \hat{v}_k, \ldots, v_n]$$

That is: the homological boundary is the $(n-1)$–chain obtained by summing the $(n-1)$–faces of σ^n, with sign. We extend ∂_n linearly to obtain a homomorphism

$$\partial_n \colon C_n(X) \to C_{n-1}(X)$$

We call this the n^{th} *boundary operator*. ◇

In the relevant dimensions (for us) we have:

$$\begin{aligned}\partial_0\sigma^0 &= 0\\ \partial_1\sigma^1 &= \sigma^1|[v_1] - \sigma^1|[v_0]\\ \partial_2\sigma^2 &= \sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]\end{aligned}$$

Exercise 8.8. Prove that $\partial_1 \circ \partial_2 \colon C_2(X) \to C_0(X)$ is the trivial homomorphism. ◇

It is a nice exercise to prove, for all $n \geq 1$, that $\partial_{n-1} \circ \partial_n = 0$. However, we will only need this fact in dimension one.

8.9. **Cycles and boundaries.**

Definition 8.10. An n–chain $c \in C_n(X)$ is an n*–cycle* if $\partial_n c = 0$. An n–chain $c \in C_n(X)$ is an n*–boundary* if there is some $d \in C_{n+1}(X)$ so that $\partial_{n+1} d = c$. ◇

The set of n–boundaries, $B_n(X)$, and the set of n–cycles, $Z_n(X)$, give subgroups of $C_n(X)$. Futhermore, because $\partial_{n-1} \circ \partial_n = 0$ we have

$$B_n(X) < Z_n(X) < C_n(X)$$

The quotient group

$$H_n(X) = Z_n(X)/B_n(X)$$

is called the "n^{th} (singular) homology group" of X. The elements of $H_n(X)$ are called "n–dimensional homology classes" in X. We will only need this in dimension $n = 1$.

There are two more pieces of terminology which are very common. Suppose that c and d are n–chains. If c is an n–boundary we may say that c is *null-homologous*. If $c - d$ is an n–boundary then we may say that c is *homologous* to d.

8.11. **Examples of cycles and boundaries.** When discussing the various groups (of chains, cycles, boundaries, and homology classes) one must always specify the ambient space X. We explore this in a collection of examples.

Example 8.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a, b] \to U$ is a closed contour. Using Lemma 8.2, let $p\colon \Delta^1 \to [a, b]$ unique affine simplex sending v_0 to p and v_1 to q. Let $c_\gamma = \gamma \circ p$. Thus c_γ is a one-chain in U. Since γ is a closed contour we find that c_γ is a one-cycle in U. ◇

In many writings the messing about (reparametrising to use Δ^1 instead of an interval of $\mathbb{R}$) is skipped. In such cases, the reader is expected to track orientations.

Example 8.13. Suppose that $T = T(p, q, r)$ is a triangle in the plane $\mathbb{C}$. Applying Lemma 8.2 we define $c\colon \Delta^2 \to T$ to be the unique affine simplex sending v_0 to p, v_1 to q, and v_2 to r. Then c is a singular two-cell, so is a two-chain. The topological boundary of T, namely $[p, q] \cup [q, r] \cup [r, p]$ (equipped with that orientation) is the image of the one-boundary ∂c. ◇

Example 8.14. Suppose that $\iota\colon \mathbb{C}^\times \to \mathbb{C}$ is the inclusion mapping. Consider the contour $\gamma\colon \Delta^1 \to \mathbb{C}^\times$ given by $\gamma(t) = e^{2\pi i x_1}$. Note that $\gamma((1, 0)) = \gamma((0, 1))$; so γ is a one-cycle in $\mathbb{C}^\times$. We define $\delta = \iota \circ \gamma$; so δ is a one-cycle in $\mathbb{C}$.

We claim that δ is a one-boundary in $\mathbb{C}$. To see this, first define $c\colon \Delta^2 \to \mathbb{C}$ by

$$c(x_0, x_1, x_2) = \begin{cases} 0, & \text{if } x_2 = 1 \\ (1 - x_2)e^{2\pi i \frac{x_1}{1 - x_2}}, & \text{if } x_2 \neq 1 \end{cases}$$

Note that c is continuous away from the vertex $v_2 = (0, 0, 1)$. To prove that c is continuous at v_2, consider the arc $I_r \subset \Delta^2$ with $x_2 = 1 - r$. So $c|I_r$ parametrises $C(0; r)$, the circle of radius r about the origin. As r tends to zero (and so I_r tends to v_2) the circle $C(0; r)$ tends to the origin. Thus small neighbourhoods of v_2 are sent to small neighbourhoods of $c(v_2)$, as desired.

Note that $c|[v_0, v_1] = \delta$ (as singular one-simplices). Also, $c|[v_0, v_2] = c|[v_1, v_2]$ (also as singular one-simplices). We deduce that $\delta = \partial c$. Thus δ is a one-boundary in $\mathbb{C}$, as desired.

On the other hand, γ is not a one-boundary in $\mathbb{C}^\times$. The proof will be asked for in Exercise 9.3 after we have Theorem 9.2 in hand. ◇

Exercise 8.15. Check the claimed equality $c|[v_0, v_2] = c|[v_1, v_2]$ in Example 8.14. ◇

9. Cauchy's theorem

Here is one of the main results of the module.

Proposition 9.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continous. Suppose that f integrates to zero along all closed contours in all balls in U. Suppose that Γ is a one-boundary in U which is piecewise C^1. Then we have:*

$$\int_\Gamma f\, dz = 0$$

Proof. By hypothesis, Γ is a one-boundary in U. Thus, by Definition 8.10 there is some finite formal sum $d = \sum_k \sigma_k^2 \in C_2(U)$ of singular two-simplices so that $\Gamma = \partial d$. Note that each σ_k^2 is continuous and Δ^2, the standard two-simplex, is compact. So $D_k = \sigma_k^2(\Delta^2)$ is compact and contained in U. By Exercise 2.5 there is a constant $\epsilon_k > 0$ so that the neighbourhood $N(D_k; \epsilon_k)$ lies in U.

We define $\epsilon = \min_k\{\epsilon_k\}$. Now there is some δ_k so that if $u, v \in \Delta^2$ have $|u - v| < \delta_k$ then $|\sigma^2_k(u) - \sigma^2_k(v)| < \epsilon$. We define $\delta = \min_k\{\delta_k\}$. Let $T^{(N)} = (\Delta^2)^{(N)}$ be the N^{th} midpoint subdivision of Δ^2. Every triangle of $T^{(N)}$ has diameter $\sqrt{2}/2^N$. Fix N so that $\sqrt{2}/2^N < \delta$.

Suppose that t is a triangle of $T^{(N)}$. We define $\sigma^2_{k,t} = \sigma^2_k|t$. So $\sigma^2_{k,t}$ is itself a singular two-simplex. We define $(\sigma^2_k)^{(N)} = \sum_t \sigma^2_{k,t}$ and we define $d^{(N)} = \sum_k (\sigma^2_k)^{(N)}$.

Define $\Gamma^{(N)}$ to be the N^{th} midpoint subdivision of Γ. Note that cancelling singular one-simplices, when subdivided, again cancel term-by-term. Thus taking boundaries commutes with midpoint subdivision; this is another version of Remark 5.13. So $\Gamma^{(N)} = (\partial d)^{(N)} = \partial(d^{(N)})$. By Lemmas 2.14 and 2.15 we have

$$\int_\Gamma f\,dz = \int_{\Gamma^{(N)}} f\,dz$$

Now we *straighten*. Suppose that t is a triangle of $T^{(N)}$. Suppose that u, v, and w are the vertices of t. We define $\tau_{k,t}$ to be the affine two-simplex mapping the vertices u, v and w to, respectively,

$$\sigma^2_{k,t}(u), \quad \sigma^2_{k,t}(v), \quad \text{and} \quad \sigma^2_{k,t}(w)$$

We call $\tau_{k,t}$ the *straightening* of $\sigma^2_{k,t}$. We lighten the notation a little bit; we define $d' = \sum_k \sum_t \tau_{k,t}$. (This is the straightening of the subdivision of d.) Let $\Gamma' = \partial d'$. (So this is the straightening of the subdivision of Γ.)

Recall that u and v are vertices of t and the latter has diameter less than δ. Also, $\tau_{k,t}$ agrees with $\sigma^2_{k,t}$ on u and v. We deduce that $\tau_{k,t}([u, v])$ lies in $B(\sigma^2_{k,t}(u); \epsilon)$. Thus $\tau_{k,t}([u, v])$ lies in U. By the same reasoning $\tau_{k,t}(t)$ lies in $B(\sigma^2_{k,t}(u); \epsilon) \subset U$. We deduce that $d' \in C_2(U)$ and so $\Gamma' \in B_1(U)$.

Now, since $\tau_{k,t}(t)$ lies in a disc $B(\sigma^2_{k,t}(u); \epsilon)$ by our hypothesis on f we have

$$\int_{\partial\tau_{k,t}} f\,dz = 0$$

Summing over k and t, we have

$$\int_{\Gamma'} f\,dz = 0$$

By construction, the non-zero terms of the affine one-boundary Γ' exactly give the straightening of the non-zero terms of $\Gamma^{(N)}$. Suppose that η' and η are corresponding terms of Γ' and $\Gamma^{(N)}$. So

- η' is an affine one-simplex while η is a piecewise C^1 one-simplex,
- the endpoints and orientations of η' and η agree, and
- the images of both are contained in the same ϵ–ball in U.

So $\eta' - \eta$ is a closed contour. Thus by the hypothesis on f we have

$$\int_{\eta' - \eta} f\, dz = 0$$

Lemma 2.14 gives

$$\int_{\eta'} f\, dz = \int_{\eta} f\, dz$$

Summing gives

$$\int_{\Gamma'} f\, dz = \int_{\Gamma^{(N)}} f\, dz$$

This completes the proof. □

As a corollary we can give the more commonly-seen statement.

Theorem 9.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that Γ is a piecewise C^1 one-boundary in U. Then we have:*

$$\int_{\Gamma} f\, dz = 0$$

Proof. Suppose that f is holomorphic. By Exercise 4.7 we have that f is continuous. By Lemma 6.3 we have that f integrates to zero around triangles in discs in U. By Lemma 7.9 we have that f has primitives in discs in U. By Lemma 7.6 we have that f integrates to zero along all closed contours in all discs in U. We now may apply Proposition 9.1 and are done. □

Exercise 9.3. With the notation as in Example 8.14: prove that γ is not a one-boundary in $\mathbb{C}^\times$. ◇

10. Triangulations

There is a tension between two of our uses of the word "boundary".

Suppose that C is a subset of the plane. The boundary of C, in the sense of point set topology and denoted ∂C, is the closure of C minus the interior of C.

On the other hand, suppose that $c = \sum a_i \sigma_i^2$ is a two-chain in the plane. In Definition 8.10 we defined the "homological boundary" $\partial c \in B_1(X)$.

Now take the union of the images of the σ_i to obtain a subset C of the plane. There is no particular reason for the simplices of ∂c to parametrise the "topological boundary" ∂C (or even to have images landing inside of ∂C).

The situation where the simplices of c nicely parametrise C, and so the simplices of ∂c nicely parametrise ∂C, is very special. Here is one possible formalisation.

Definition 10.1. Suppose that C is a closed, bounded, path-connected subset of the plane with non-empty interior. Suppose that ∂C (equipped with the subspace topology) is homeomorphic to a collection of circles.

Suppose that $c = \sum \sigma_k^2$ is a two-chain in C. We take one copy of Δ^2 for each k; call it Δ_k^2. We say that c gives an *edge pairing* of an edge e of Δ_k^2 with an edge f of Δ_ℓ^2 if $\sigma_k^2|e = -\sigma_\ell^2|f$ as singular one-simplices. (That is, the functions $\sigma_k^2|e$ and $\sigma_\ell^2|\bar{f}$ are equal, where $\bar{f}$ is the reverse of f.) We form a topological space X_c by gluing the two-simplices Δ_k^2 according to the edge parings.

We form a continuous function $\Sigma_c \colon X_c \to C$ by defining $\Sigma_C(x) = \sigma_k(x)$ if x lies in Δ_k^2. This is well-defined by the construction of X_c.

We say that c *triangulates* C if $\Sigma_c \colon X_c \to C$ is a homeomorphism. ◇

We make a similar definition for one-chains triangulating contours. Only very special regions in the plane can be triangulated; nonetheless such regions are extremely important to the theory.

Example 10.2. Suppose that R is a rectangle in the plane. Suppose that p, q, r, and s are the vertices of R, in that anticlockwise order. We pick a diagonal for R, say $[p, r]$. This divides R into a pair of triangles Q and S. Using Lemma 8.2 we obtain affine two-simplices σ_Q and σ_S so that

- σ_Q sends v_0, v_1, and v_2 to q, r, and p (in that order) and
- σ_S sends v_0, v_1, and v_2 to s, p, and r (in that order).

Then the two-chain $c = \sigma_Q + \sigma_S$ triangulates R; also, ∂c triangulates ∂R. ◇

Definition 10.3. Suppose that $0 < a < b$ is a pair of real numbers. We define

$$A(z_0; a, b) = \{z \in \mathbb{C} \mid a < |z - z_0| < b\}$$

This is the *(open) annulus* centred on z_0, with inner radius a, and with outer radius b. ◇

It is convenient to extend the notation of Definition 10.3 as follows.

- $A(z_0; 0, b) = B^\times(z_0; b)$ is a punctured ball.
- $A(z_0; a, \infty) = \mathbb{C} - \overline{B(z_0; a)}$ is the complement of a closed ball.
- $A(z_0; 0, \infty) = \mathbb{C}^\times$ is the punctured plane.

Example 10.4. Here is another example, which will be important for our later work. Suppose that $0 < a < b$ is a pair of real numbers Set $A = \overline{A(0; a, b)}$: the closure of an open annulus.

Suppose that R is the closed rectangle in the plane with vertices at

$$\log(a), \quad \log(b), \quad \log(b) + 2\pi i, \quad \log(a) + 2\pi i$$

Suppose that $d \in C_2(R)$ is a two-chain which triangulates R, as in Example 10.2. Let $c = \textsc{Exp} \circ d$ be the two-chain obtained by post-composing the singular simplices of d with the complex exponential.

Recall (Exercise 4.21) that EXP sends horizontal segments of R to radial segments of the annulus A; in particular the bottom and top of R are both sent to the segment $[a, b]$ in the real line. Also, EXP sends vertical segments of R to circles (centred about the origin) in A. Thus R parametrises A and so c triangulates A. ◇

When carrying out certain contour integrals, we will need to triangulate more general regions.

Lemma 10.5. *Suppose that C is a connected closed compact subset of the plane with non-empty interior. Suppose that ∂C is a finite union of simple closed contours. Suppose further that every point p of ∂C lies in the closure of the interior of C. Then there is a two-chain which triangulates C.*

Proof. Since ∂C is a contour, it comes to us equipped with a parametrisation. Fix a point $p \in \partial C$ which is not a corner. Pick a small subarc γ of ∂C, centred on p: that is, $\gamma(0) = p$. We pick γ sufficiently small so that the tangent vectors along γ are almost parallel to $\gamma'(0)$ and have almost the same length. We now pick an even smaller square S_p, centred on p, so that:

- S_p has two "horizontal" sides parallel to $\gamma'(0)$,
- γ meets S_p in a single connected arc,
- the intersection $\gamma \cap S_p$ is a graph over one (thus both) of the horizontal sides, and
- γ enters and exits S_p through the interiors of the remaining two "vertical" sides.

Note that ∂C may enter S_p several times. Applying ideas from the proof of Exercise 2.5, we find that any other components of ∂C running through S_p stay a definite distance away from $\gamma \cap S_p$. So we can shrink S_p, maintaining the above properties, to ensure that S_p meets ∂C in a single connected arc with the above properties. (A similar argument gives a parallelogram for each corner.)

As p ranges over ∂C the squares (and parallelograms) S_p cover ∂C. So there is a finite subcollection that again covers ∂C. Shrinking these as needed, we can assume that every square (or parallelogram) meets exactly two others: one before and one after. We slightly extend all of the vertical sides of all of the squares (and parallelograms) as follows: Each component of ∂C, cut by all of the vertical sides, is contained in a quadrilateral. These quadrilaterals form annulus neighbourhoods of the components of ∂C. Each component of ∂C cuts its annulus in two – one subannulus lies inside of C and the other lies outside.

In turn, the arcs of ∂C divide each quadrilateral into two almost quadrilaterals – one inside and one outside of C. ("An almost quadrilateral", because one of the sides is a subarc of ∂C.) Finally, we use the fact that $\partial C \cap S_p$ is a graph over the horizontal sides in order to cut the almost quadrilateral into a pair of parametrised triangles.

We extend all linear sides of all almost quadrilaterals into C (away from the annuli about ∂C). This cuts what remains into convex polygons. We triangulate these as in the solution to Exercise 7.2. □

Remark 10.6. In Definition 5.7 and also Examples 10.2 and 10.4 we trianglated various regions C: respectively, a triangle, a rectangle, and an annulus. In each case there was a two-chain d (consisting of one triangle or two). We ordered the vertices of the triangles in the two-chain so that all triangles were positively oriented (anticlockwise, according to Definition 5.7). In the case of the triangle and the rectangle, we find that ∂d triangulates ∂C and gives it an anticlockwise orientation.

In the case of the annulus, we push d forward by the exponential map. Thus gives the two-chain c, which triangulates $C = \overline{A(0; a, b)}$. We deduce that the homological boundary ∂c parametrises the "topological boundary" ∂C. This is now a union of two circles: $\partial_a C$ (inner) and $\partial_b C$ (outer). Given our choice of d the induced orientations on $\partial_a C$ and $\partial_b C$ are, respectively, clockwise and anticlockwise.

In general, suppose that C is a region as in Lemma 10.5. The lemma gives us a triangulation with all triangles oriented positively; thus all of their boundaries are oriented anticlockwise. These orientations induced orientations on the components of ∂C. Suppose that γ is one such. Then the induced orientation on γ is clockwise and anticlockwise as γ is an "interior" or the "exterior" component of ∂C.

Here is one way to remember our orientation conventions. Again take γ to be a component of ∂C. We imagine ourselves standing on the plane (with our head above the plane) with our feet on γ. We turn around (if needed) to ensure that C is to our *left*. This done, we are facing in the direction of the induced orientation on γ. ◇

Exercise 10.7. Go outside. Find a region C on the ground. Pick a component γ of ∂C. Run around γ in the direction of its induced orientation. ◇

11. Cauchy's integral theorem

Theorem 9.2 is one of the highlights of Cauchy's theory of contour integrals. We can use it to quickly prove the following "global-to-local" result.

Theorem 11.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $r > 0$ is a radius so that $\overline{B(z_0; r)}$ is contained in U. Let $C = C(z_0; r)$ be the boundary of $\overline{B(z_0; r)}$. Suppose that w lies in $B(z_0; r)$. Then we have:*

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} \, dz$$

Proof. Pick $\epsilon > 0$ so that $\overline{B(w;\epsilon)}$ is contained in $B(z_0;r)$. Let C_ϵ be the boundary of $\overline{B(w;\epsilon)}$; so C_ϵ is contained in $B(z_0;r)$ and thus is disjoint from C. We orient both C and C_ϵ anticlockwise.

We take $A_\epsilon = \overline{B(z_0;r)} - B(w;\epsilon)$. So A_ϵ is closed, is homeomorphic to an annulus, and has $\partial A_\epsilon = C \sqcup C_\epsilon$. Applying Lemma 10.5 we obtain a two-chain $d \in C_2(U)$ that triangulates A_ϵ. Thus $\Gamma = \partial d$ triangulates, and orients, C and C_ϵ. As in Remark 10.6 we arrange matters so that C is oriented anticlockwise by Γ and C_ϵ is oriented clockwise by Γ. Thus, in a small abuse of notation justified by Lemmas 2.14 and 2.15, we write $\Gamma = C - C_\epsilon$. (An "abuse" because Γ and $C - C_\epsilon$ are not equal but rather are homologous one-chains.)

Note that C and C_ϵ are circles, so are both contours (that is, C^1). By Theorem 9.2 we have that

$$\int_\Gamma \frac{f(z)}{z-w}\,dz = 0$$

Thus

$$\int_C \frac{f(z)}{z-w}\,dz = \int_{C_\epsilon} \frac{f(z)}{z-w}\,dz$$

To understand the latter recall, from Lemma 4.8, that we can write

$$f(z) = f(w) + f'(w)(z-w) + \rho(z)(z-w)$$

with ρ continuous in U (and vanishing at w). Let M be the maximum of $|\rho(z)|$ as z ranges over $\overline{B(z_0;r)}$. We now compute as follows.

$$\begin{aligned}
&\left| \left[\int_{C_\epsilon} \frac{f(z)}{z-w}\,dz \right] - 2\pi i f(w) \right| = \\
&= \left| \int_{C_\epsilon} \frac{f(z)}{z-w}\,dz - f(w) \int_{C_\epsilon} \frac{1}{z-w}\,dz \right| && \text{Example 2.12} \\
&= \left| \int_{C_\epsilon} \frac{f(z)-f(w)}{z-w}\,dz \right| && w \text{ is independent of } z \\
&= \left| \int_{C_\epsilon} (f'(w) + \rho(z))\,dz \right| && \text{Lemma 4.8} \\
&\le (|f'(w)| + M) \cdot 2\pi\epsilon && \text{Lemma 2.22}
\end{aligned}$$

Since $\epsilon > 0$ is as small as we like, and independent of w and M, this completes the proof. □

11.2. **The mean value theorem.** Here is a useful special case of Theorem 11.1.

Theorem 11.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $r > 0$ is a radius so that $\overline{B(z_0;r)}$ is contained in U. Then we have:*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

In words: the value of a holomorphic function at a point is the average of its values on any circle about that point.

Proof of Theorem 11.3. In Theorem 11.1 we take $w = z_0$ and we parametrise $C = C(z_0; r)$ by $\gamma(\theta) = z_0 + re^{i\theta}$. We now compute as follows:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta) - z_0} \gamma'(\theta)\, d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \end{aligned}$$ □

12. Analytic functions

12.1. **From holomorphic to analytic.**

Definition 12.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f : U \to \mathbb{C}$ is a function. Suppose that for all $z_0 \in U$ there is some $r > 0$ so that

- $B(z_0; r) \subset U$ and
- f agrees with some power series in $B(z_0; r)$.

Then we call f *analytic.* ◇

From Theorem 4.13 we deduce the following.

Corollary 12.3. *Analytic functions are holomorphic.* □

Our goal in this section is to prove the converse.

Theorem 12.4. *Holomorphic functions are analytic.*

Proof. Here are our standing hypotheses and notations. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f : U \to \mathbb{C}$ is holomorphic. Fix a point z_0 in U. Fix r so that the closed ball $\overline{B(z_0; r)}$ lies in U. Let $C = C(z_0; r)$ be the boundary of $\overline{B(z_0; r)}$.

We define, for $n \in \mathbb{N}$:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

(By Theorem 11.1 we have $a_0 = f(z_0)$.) Set

$$M = \max\left\{ |f(w)| : w \in \overline{B(z_0; r)} \right\}$$

Lemma 12.5. $|a_n| \leq M/r^n$

Exercise 12.6. Prove Lemma 12.5. ◇

We deduce that the power series $F(w) = \sum_{n=0}^{\infty} a_n(w - z_0)^n$ converges in the ball $B(z_0; r)$.

Lemma 12.7. *For all $w \in B(z_0; r)$ we have $f(w) = F(w)$.*

Proof. We compute as follows:

$$
\begin{aligned}
F(w) &= \sum_{n=0}^{\infty} a_n(w - z_0)^n && \text{definition } F(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right] (w - z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z - z_0)^{n+1}} (w - z_0)^n \right] dz && \text{interchange} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\sum \left(\frac{w - z_0}{z - z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\frac{1}{1 - \frac{w - z_0}{z - z_0}} \right] dz && \text{geometric series} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - w} \, dz && \text{simplify} \\
&= f(w) && \text{Theorem 11.1}
\end{aligned}
$$

The uniform convergence of the power series justifies the interchange of sum and integral. The geometric series $\sum \left(\frac{w-z_0}{z-z_0}\right)^n$ converges because w lies in the interior of $\overline{B(z_0; r)}$ while z lies on C, the boundary of $\overline{B(z_0; r)}$. □

This completes the proof of Theorem 12.4 □

We call $F(w)$, the power series appearing in the proof, the *series expansion of f about z_0*. Exercise 4.15 proves that F is unique, justifying the name.

Exercise 12.8. Give a direct proof that the interchange of sum and integral, in the proof of Theorem 12.4, is permitted. ◇

The proof of Theorem 12.4 gives an integral expression for the coefficients a_n of the series expansion. This, and Lemma 12.5, gives the following.

Corollary 12.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f: U \to \mathbb{C}$ is holomorphic. Suppose that $z_0 \in U$. Let R be the radius of convergence of the series expansion of f about z_0. Suppose that $B(z_0; r)$ is a subset of U. Then $r \leq R$.* □

In particular, if $U = \mathbb{C}$ then the radius of convergence is infinite. The converse does not quite hold.

12.10. **The fundamental theorem of complex analysis.** We gather our results so far into an "omnibus", inspired by [8, Theorem 1.1, page 3]. See [6, Theorem 8.2.1, page 236] for further equivalent properties.

Theorem 12.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then the following are equivalent.*

- *f is holomorphic in U*
- *f integrates to zero about triangles in U*
- *f has a primitive in all balls in U*
- *f integrates to zero about all closed contours in all balls in U*
- *f integrates to zero about all one-boundaries in U which are piecewise C^1*
- *for all $z_0 \in U$ and $r > 0$ with $\overline{B(z_0; r)} \subset U$, for all $w \in B(z_0; r)$, and taking C to be the boundary of $\partial\overline{B(z_0; r)}$, we have*
$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w}\, dz$$
- *f is analytic in U and, at a point $z_0 \in U$, the n^{th} coefficient of its series expansion is*
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$
where $C = C(z_0; \epsilon)$ lies in U. □

13. Zeros

13.1. **Orders of zeros.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic.

We have two pieces of "global" terminology.

Definition 13.2. Suppose that $f(z) = 0$ for all $z \in U$. Then we say that f *vanishes identically* in U.

On the other hand, suppose that $f(z) \neq 0$ for at least one $z \in U$. Then we say that f is not *identically zero* in U. ◇

We also have two bits of "local" terminology.

Definition 13.3. Suppose that z_0 is a point of U. Suppose that $A = (a_n)_n$ are the coefficients of the series expansion of f about z_0, as provided by Theorem 12.11.

Suppose that all of the a_n are zero. Then we say that f *vanishes to infinite order* at z_0.

On the other hand, suppose that not all of the a_n are zero. So there is a least N so that $a_N \neq 0$. We call N the *order of vanishing* of f at z_0; we denote this natural number N by $\textsc{Ord}(f, z_0)$. If the order is positive, then we call z_0 a *zero* of f of *order* $\textsc{Ord}(f, z_0)$. If the order is one, then we call z_0 a *simple zero* of f. ◇

A point $z_0 \in U$ where $\textsc{Ord}(f, z_0) = 0$ is sometimes called an *ordinary* or a *regular* point of f.

Lemma 13.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$. Suppose that f vanishes to infinite order at $z_0 \in U$. Then f vanishes identically in U.*

Proof. Suppose that z_1 is another point of U.

Exercise 13.5. Prove that there is a constant $\epsilon > 0$ and a sequence of points $(w_k)_{k=0}^N \subset U$ as follows.

- $w_0 = z_0$,
- $w_N = z_1$,
- for all k we have $|w_{k+1} - w_k| < \epsilon$, and
- for all k the distance from w_k to any point of $\mathbb{C} - U$ is at least ϵ. ◇

With the points $(w_k)_{k=0}^N$ in hand, we induct. Applying Theorem 12.11 and Exercise 4.15, we take A_k to be the coefficients of the series expansion of f at w_k. By hypothesis, A_0 is the zero sequence. Suppose that, by induction, we have that A_k is also the zero sequence. Since the constant term is zero, we in particular have that $f(w_k) = 0$. If $k = N$ then $w_k = w_N = z_1$ and so $f(z_1) = 0$; in this case we are done. Suppose that $k < N$. Since A_k is the zero sequence, Theorem 4.13 tells us that f vanishes identically in $B(w_k; \epsilon)$. Note that w_{k+1} lies in $B(w_k; \epsilon)$. Thus all derivatives of f vanish at w_{k+1}. Applying Theorem 4.13 again, we find that A_{k+1} is also the zero sequence. This completes the induction step and the proof. □

13.6. **Patching.**

Lemma 13.7. *Suppose that U and V are domains in $\mathbb{C}$ with non-empty intersection. Then $U \cup V$ is again a domain in $\mathbb{C}$.* □

We can now state the "patching" lemma.

Lemma 13.8. *Suppose that U and V are domains with non-empty intersection. Suppose that $f\colon U \to \mathbb{C}$ and $g\colon V \to \mathbb{C}$ are holomorphic. Suppose further that $f|(U \cap V) = g|(U \cap V)$. Then there is a unique holomorphic function $h\colon U \cup V \to \mathbb{C}$ so that $h|U = f$ and $h|V = g$.*

Proof. We define h piecewise by

$$h(z) = \begin{cases} f(z), & z \in U \\ g(z), & z \in V \end{cases}$$

Our hypothesis implies that h is well-defined.

Suppose that z_0 is a point of $U \cup V$. Thus z_0 lies in U or in V. As the other case is similar, suppose that z_0 lies in U. Thus there is some $\epsilon > 0$ so that $B = B(z_0; \epsilon)$ is a subset of U. Since $h|B = f|B$ we find that $h'(z_0) = f'(z_0)$. Thus h has a complex derivative at every point of $U \cup V$. So h is holomorphic on $U \cup V$. □

Remark 13.9. The conclusion of the proof relies on the fact that holomorphicity is a *local property*: that is, depends only on properties knowable from behaviour in arbitrarily small neighbourhoods of points. ◇

The patching lemma (13.8) is a holomorphic version of the *gluing lemma* in point-set topology: the latter builds continuous functions from continuous pieces.

13.10. **Factoring.**

Lemma 13.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not identically zero in U. Fix $z_0 \in U$. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $g(z_0) \neq 0$ and so that*

$$f(z) = (z - z_0)^{\textsc{Ord}(f,z_0)} g(z)$$

Proof. Applying Lemma 13.4 we find that $\textsc{Ord}(f, z_0)$ is finite; suppose that f vanishes to order $\textsc{Ord}(f, z_0) = N > 0$ at z_0. Since f is holomorphic in U it has a series expansion about with some radius $R > 0$ of convergence. Suppose that $(a_n)_n$ are the coefficients; thus $a_0 = a_1 = \ldots = a_{N-1} = 0$ while $a_N \neq 0$. Let $B = (b_n)_n$ be given by $b_n = a_{n+N}$. Appealing to Theorem 4.13 we define $g_0\colon B(z_0; R) \to \mathbb{C}$ to be the holomorphic function with power series with coefficients B. By Theorem 4.14 we have $f(z) = (z - z_0)^N g_0(z)$ on $B(z_0; R)$.

We next define $g_1\colon U - \{z_0\} \to \mathbb{C}$ by $g_1(z) = (z - z_0)^{-N} f(z)$. This is holomorphic by the quotient rule. Note that g_0 and g_1 agree on the punctured ball $B(z_0; R) - \{z_0\}$. Applying Lemma 13.8 gives the desired holomorphic $g\colon U \to \mathbb{C}$. □

Exercise 13.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $g, f\colon U \to \mathbb{C}$ are holomorphic. Suppose that z_0 is a point of U. Prove that $\textsc{Ord}(fg, z_0) = \textsc{Ord}(f, z_0) + \textsc{Ord}(g, z_0)$ (where we allow either or both of f and g to vanish to infinite order at z_0). ◇

14. The identity theorem

14.1. **Isolated sets.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$. A point $w \in U$ is an *accumulation point* of Z if, for all $\epsilon > 0$, the punctured ball $B(w; \epsilon) - \{w\}$ has non-empty intersection with Z. If Z has no accumulation points in U then we say that Z is *isolated* in U.

Suppose that Z is isolated in U. The definition implies that Z is closed in U. Furthermore, for all $z_0 \in Z$, there is an $r_0 > 0$ so that

- $B(z_0; r_0) \subset U$ and
- $B(z_0; r_0) \cap Z = \{z_0\}$.

The converse does not hold, as shown by the following.

Example 14.2. Suppose that $Z \subset \mathbb{C}$ is the set

$$Z = \{i/n \mid n \in \mathbb{N}, n > 0\}$$

For $z_n = i/n$ we can take the disc $B_n = B(z_n, 1/(n^2 + n))$ and find that $B_n \cap Z = \{z_n\}$. Nonetheless, Z is not isolated in $\mathbb{C}$; it has an accumulation point at the origin.

On the other hand, Z is isolated in the punctured plane $\mathbb{C}^\times$. (Or in any domain U containing Z but not containing the origin.) ◇

Exercise 14.3. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Suppose that $K \subset U$ is compact. Prove that $Z \cap K$ is finite. ◇

Exercise 14.4. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Then so is any subset of Z.

Suppose additionally that $Z' \subset U$ is isolated in U. Then so is $Z \cup Z'$. ◇

Exercise 14.5. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is isolated in U. Then $U - Z$ is again a domain. ◇

14.6. **Zeros are isolated.**

Lemma 14.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not identically zero in U. Then the set of zeros $Z(f)$ is isolated in U.*

Proof. Fix z_0 in U. Suppose that $N = \textsc{Ord}(f, z_0)$. Lemma 13.11 gives us a holomorphic function $g\colon U \to \mathbb{C}$ so that

- $g(z_0) \neq 0$ and
- $f(z) = (z - z_0)^N g(z)$.

By Exercise 4.7 we have that g is continuous. Thus there is a small disc $B(z_0; r) \subset U$ where g does not vanish. If $N > 0$ then $(z - z_0)^N$ vanishes only at z_0; in $N = 0$ then $(z - z_0)^0$ is the constant function giving 1. In either case, the product $(z - z_0)^N g(z)$ has no zeros in the punctured ball $B(z_0; r) - \{z_0\}$. □

Example 14.8. Take $U = \mathbb{C}^\times$ to be the punctured plane. Define $f\colon U \to \mathbb{C}$ by $f(z) = \textsc{Sin}(1/z)$. By Exercise 4.20 we find

$$Z(f) = \{1/(k\pi) \mid k \in \mathbb{Z}, k \neq 0\}$$

So $Z(f)$ is isolated in U, but not in $\mathbb{C}$. ◇

14.9. **Proving the identity theorem.**

Theorem 14.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $Z \subset U$ is closed in U and has an accumulation point in U. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic. Suppose that $f|Z = g|Z$. Then we have $f = g$.*

Proof. The function $h = f - g$ is holomorphic and vanishes identically on Z. Thus, by Lemma 14.7 we have that h is identically zero in U. So $f = g$, as desired. □

15. Singularities

Singularities fall into three classes: removable singularities, poles, and essential singularities.

15.1. **Removing removable singularities.** Suppose that X is a set. Suppose that $Y \subset X$ is a subset. Suppose that f is a function on Y and F is a function on X. Suppose that $F|Y = f$. Then we call F an *extension* of f (and we call f are *restriction* of F).

As a further bit of terminology we have the following. Suppose that X is a topological space and Y is a subspace. Suppose that F and f are continuous. Then we call F a *continuous extension* of f. We similarly speak of *holomorphic extensions* and so on.

In this new language, the patching lemma (13.8) says that h is a holomorphic extension both of f and of g.

The *removable singularity theorem* tells us when a function "missing a point" has a holomorphic extension over that point. For our proof we mostly follow [6, page 212].

Theorem 15.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $U^\times = U - \{z_0\}$. Suppose that $f \colon U^\times \to \mathbb{C}$ is holomorphic. The following are equivalent.*

- *f has a holomorphic extension to U.*
- *f has a continuous extension to U.*
- *there is an $\epsilon > 0$ so that $B(z_0; \epsilon)$ is contained in U and f is bounded on $B^\times = B(z_0; \epsilon) - \{z_0\}$.*
- $\lim_{z \to z_0} (z - z_0) f(z) = 0$

Furthermore, when they exist, holomorphic extensions are unique.

Proof. The forward implications are immediate. So we must prove that the last implies the first. We define $h \colon U \to \mathbb{C}$ by:

$$h(z) = \begin{cases} 0, & \text{if } z = z_0 \\ (z - z_0)^2 f(z), & \text{if } z \neq z_0 \end{cases}$$

Note that h is holomorphic on $U^\times$.

We compute $h'(z_0)$ as follows.

$$h'(z_0) = \lim_{z \to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

Thus h is holomorphic in U.

By Theorem 12.11, we find that h has a series expansion in some ball about z_0: say $B(z_0; r)$. Suppose that (a_n) are the coefficients of the series expansion. If all of the a_n are zero, then h is vanishes identically in U (Lemma 13.4). Thus f vanishes identically in $U^\times$. So the desired holomorphic extension is the zero function.

Suppose that $N = \mathrm{O\scriptsize RD}(h, z_0)$ is finite. Recall that $h(z_0) = h'(z_0) = 0$. Thus $a_0 = a_1 = 0$ and so $N \geq 2$. We now apply Lemma 13.11 to obtain a holomorphic function $g\colon U \to \mathbb{C}$ so that $g(z_0) \neq 0$ and

$$h(z) = (z - z_0)^N g(z)$$

Thus $(z - z_0)^{N-2} g(z)$ is a holomorphic extension of f.

Finally, the extension is unique by Theorem 14.10. □

15.3. **Laurent series.** Before dealing with poles and essential singularities, we must generalise power series.

Definition 15.4. Suppose that z_0 is a complex number. Suppose that $(a_k)_{k\in\mathbb{Z}}$ is a bi-infinite sequence of complex numbers. Then we write

$$\sum_{k\in\mathbb{Z}} a_k (z - z_0)^k$$

for the *Laurent series* with *centre* z_0 and *coefficients* (a_k). We define a pair of radii as follows.

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$
$$r = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}$$

We call these the *outer* and *inner* radii of convergence, respectively. As in Definition 4.12, these radii can be zero or infinity. We call the term a_{-1} the *residue* of the series. We call the subsum $\sum_{k<0} a_k(z - z_0)^k$ the *principal part* of the series. ◇

Exercise 15.5. Following the notation of Definition 15.4, suppose that the inner radius r is strictly less than the outer radius R. Prove that the sum

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

converges uniformly on compact subsets of $U = A(0; r, R)$. Also, prove that the resulting function is holomorphic. ◇

Just as Laurent polynomials (Exercise 2.16) generalise ordinary polynomials, Laurent series generalise power series. By considering partial sums, we can obtain versions of Theorems 4.13 and 4.14 for Laurent series. There is also a converse of Exercise 15.5 – that is, a version of Theorem 12.11. Here is one possible statement.

Theorem 15.6. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of $\mathbb{C}$. Suppose that $A = A(z_0; a, b)$ is an annulus whose closure is contained in U.*

Suppose that $a \leq r \leq b$. Suppose that $C = C(z_0; r)$. We define, for all n in $\mathbb{Z}$, the coefficient

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

Then the Laurent series $\sum a_n (z - z_0)^n$ converges uniformly to f on the closure of A. □

With f as in the theorem, we call a_{-1} the *residue* of f at z_0; we denote this by $\mathrm{RES}(f, z_0)$.

15.7. **Poles.** Here is an immediate corollary of Theorem 15.2.

Corollary 15.8. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $U^\times = U - \{z_0\}$. Suppose that $f \colon U^\times \to \mathbb{C}$ is holomorphic. Suppose that $N \geq 0$ is a non-negative integer. Suppose that*

$$\lim_{z \to z_0} (z - z_0)^{N+1} f(z) = 0$$

Then $(z - z_0)^N f(z)$ has a (unique) holomorphic extension to U. □

Applying Theorem 12.11, the holomorphic extension of $(z - z_0)^N f(z)$ has a series expansion in some ball $B = B(z_0; r) \subset U$. Suppose that (c_n) are the resulting coefficients. Then

$$\sum_{n=0}^{\infty} c_n (z - z_0)^{n-N} = \sum_{m=-N}^{\infty} c_{m+N} (z - z_0)^m$$

is the Laurent expansion of f in the punctured ball $B^\times = B - \{z_0\}$. We define $a_m = c_{m+N}$. When $N > 0$ and $a_{-N} \neq 0$ we say that f has a *pole of order* N at z_0; in this case we write $\mathrm{ORD}(f, z_0) = -N$. We may call a pole of order one a *simple pole.*

15.9. **Essential singularities.**

Definition 15.10. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $z_0 \in U$. Set $U^\times = U - \{z_0\}$. Suppose that $f \colon U^\times \to \mathbb{C}$ is holomorphic. Suppose that principal part of the Laurent expansion of f about z_0 has infinitely many terms. Then we say that f has an *essential singularity* at z_0. ◇

Example 15.11. The function

$$\mathrm{EXP}(1/z) = \sum_{n \in \mathbb{N}} \frac{z^{-n}}{n!}$$

has an essential singularity at the origin. ◇

Theorem 15.6 gives us the following trichotomy.

Lemma 15.12. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $f\colon U^\times \to \mathbb{C}$ is holomorphic. Then z_0 is either a removable singularity, a pole of finite order, or an essential singularity of f.* □

Holomorphic functions are wild in neighbourhoods of an essential singularity, justifying the name of the latter. The following result is often called the *Casorati–Weierstrass theorem*; for a short history see [6, page 309].

Theorem 15.13. *Suppose that U, z_0, f are as in Definition 15.10. Suppose that $V^\times$ is a punctured neighbourhood of z_0. Then the image $f(V^\times)$ is dense in $\mathbb{C}$.*

Proof. We prove the contrapositive. Suppose that $f(V^\times)$ is not dense in $\mathbb{C}$. Fix $w_0 \in \mathbb{C}$ and $\epsilon > 0$ so that $f(V^\times) \cap B(w_0; \epsilon) = \varnothing$. Define $g\colon U^\times \to \mathbb{C}$ by $g(z) = 1/(f(z) - w_0)$. So $g\colon V^\times \to \mathbb{C}$ is holomorphic. Note that, for all $z \in V^\times$ we have $|f(z) - w_0| \geq \epsilon$. Thus

$$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\epsilon}$$

We deduce that g is bounded in $V^\times$. Theorem 15.2 implies that g has a removable singularity at z_0.

Let $G\colon V \to \mathbb{C}$ be the resulting holomorphic extension of g. Suppose that $N = \textsc{Ord}(G, z_0)$. Lemma 13.11 implies that there is a holomorphic function $H\colon V \to \mathbb{C}$ so that $G(z) = (z-z_0)^N H(z)$ and so that $H(z_0) \neq 0$. So, on $V^\times$, we have

$$(z - z_0)^N H(z) = \frac{1}{f(z) - w_0}$$

We deduce that on $V^\times$ we have

$$f(z) = \frac{1}{(z - z_0)^N H(z)} + w_0$$

Now, $1/H$ is holomorphic in some (perhaps smaller) neighbourhood W of z_0 by Theorem 4.14. So z_0 is either a removable singularity or a pole of finite order of f. By trichotomy (15.12) the point z_0 is not an essential singularity of f. □

16. Meromorphic functions

We use the notation $\hat{\mathbb{C}}$ to represent the *extended plane*: the set $\mathbb{C} \cup \{\infty\}$. We give $\hat{\mathbb{C}}$ a topology and a geometry in Section 24.

16.1. The definition.

The definition of a meromorphic function is unfortunately somewhat complicated. We give a simpler, equivalent, formulation in Remark 16.7; however the proof of the equivalence is beyond the methods of these lecture notes.

Definition 16.2. Suppose that $U \subset \mathbb{C}$ is a domain. We say that $h\colon U \to \hat{\mathbb{C}}$ is a *meromorphic* function if we have the following.

- The preimage $P(h) = h^{-1}(\infty)$ is isolated in U.
- The restriction $h|(U - P(h))$ is holomorphic.
- For every $z_k \in P(h)$ there is an integer $n_k > 0$ and positive $\epsilon_k > 0$ as follows:
  - $B_k^\times = B(z_k; \epsilon_k) - \{z_k\}$ is contained in $U - P(f)$ and
  - h, restricted to $B_k^\times$, has a pole of order n_k at z_k.

We call $P(h)$ the set of *poles* of h. ◇

Remark 16.3. Note that the constant function $h(z) = 0$ is meromorphic. However, the constant function $g(z) = \infty$ is not. ◇

16.4. **Examples of meromorphic functions.**

Example 16.5. The function $f(z) = 1/z$ is meromorphic in the plane. It has a simple pole (that is, of order one) at the origin and no zeros. ◇

More generally, suppose that $p, q \in \mathbb{C}[z]$ are polynomials with neither equal to the zero polynomial. Then the rational function $h = p/q$ is meromorphic in the plane. Then we have the following:

$$\textsc{Ord}(h, z_0) = \textsc{Ord}(p, z_0) - \textsc{Ord}(q, z_0)$$

Instead of proving this, we give a generalisation.

Lemma 16.6. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic functions with neither identically zero in U. Then $h = f/g$ determines a meromorphic function in U; its sets of zero and poles are*

$$Z(h) = \{w \in U \mid \textsc{Ord}(f, w) > \textsc{Ord}(g, w)\}$$
$$P(h) = \{w \in U \mid \textsc{Ord}(f, w) < \textsc{Ord}(g, w)\}$$

and for all $w \in U$ we have

$$\textsc{Ord}(h, w) = \textsc{Ord}(f, w) - \textsc{Ord}(g, w)$$

Proof. Fix $z_0 \in U$. Set $k = \textsc{Ord}(f, z_0)$ and $\ell = \textsc{Ord}(g, z_0)$. We apply Lemma 13.11 to both f and g in order to rewrite them as

$$f(z) = (z - z_0)^k F(z) \quad \text{and} \quad g(z) = (z - z_0)^\ell G(z)$$

Here F and G are holomorphic, and non-zero, in a small neighbourhood of z_0. Thus F/G is also holomorphic near z_0.

We break the proof into cases and deal with each in turn.

- Suppose that $\ell = \textsc{Ord}(g, z_0) = 0$. So, locally, $h = (z - z_0)^k F/G$ is holomorphic. In this case $h(z_0)$ is zero exactly when $f(z_0)$ is, and $\textsc{Ord}(h, z_0) = \textsc{Ord}(f, z_0)$.
- Suppose that $\ell = \textsc{Ord}(g, z_0) > 0$ There are now three possibilities.

– Suppose that $k > \ell$. We cancel $(z - z_0)^\ell$ from the top and bottom. Applying Theorem 15.2 we find that h has a removable singularity at z_0. The unique extension (again called h) has a zero at z_0 of order $k - \ell$.
– Suppose that $k = \ell$. We cancel $(z - z_0)^\ell$ from the top and bottom. Applying Theorem 15.2 we find that h has a removable singularity at z_0. The unique extension (again called h) has $h(z_0) = F(z_0)/G(z_0) \neq 0$. Thus $\textsc{Ord}(h, z_0) = 0$.
– Suppose that $k < \ell$. We cancel $(z - z_0)^k$ from the top and bottom. Thus we may write $h(z) = (z - z_0)^{k-\ell} F(z)/G(z)$. Thus h has a pole, of order $\ell - k$, at z_0.

Along the way we have produced all zeros and poles of h; this finishes the proof. □

As an application, the trigonometric function $\textsc{Tan}(z) = \textsc{Sin}(z)/\textsc{Cos}(z)$ is meromorphic in $\mathbb{C}$.

Remark 16.7. For a domain U in $\mathbb{C}$, the converse of Lemma 16.6 holds: *every* meromorphic function in U is obtained as a ratio of holomorphic functions in U. This follows from Weierstrass's *product theorem*. We will not pursue this line of inquiry. ◇

16.8. **Patching, factoring, isolating.** We now generalise a few results from holomorphic to meromorphic functions.

Lemma 16.9. *Suppose that U and V are domains with non-empty intersection. Suppose that $f\colon U \to \hat{\mathbb{C}}$ and $g\colon V \to \hat{\mathbb{C}}$ are meromorphic. Suppose further that $f|(U \cap V) = g|(U \cap V)$. Then there is a unique meromorphic function $h\colon U \cup V \to \hat{\mathbb{C}}$ so that $h|U = f$ and $h|V = g$.*

This must hold, because meromorphicity is a local property.

Proof of Lemma 16.9. Since U and V intersect, their union $W = U \cup V$ is a domain.

Note that $P(f)$ is isolated in U and $P(g)$ is isolated in V. Since f and g agree on $U \cap V$ we have that $P(f) \cap V = P(g) \cap U$. Applying Exercise 14.4 we have that $P(f) \cap V$ is isolated in V. Suppose that z_0 is a point of V. Fix $\epsilon > 0$ with $B = B(z_0; \epsilon) \subset V$. So z_0 is an accumulation point of $P(f)$ if and only if it is an accumulation point of $P(f) \cap B \subset P(f) \cap V$. So z_0 is not an accumulation point of $P(f)$. We deduce that $P(f)$ is isolated in $U \cup V$. Similarly, $P(g)$ is isolated in $W = U \cup V$. Applying Exercise 14.4 we find that $P(f) \cup P(g)$ is isolated in W.

By Exercise 14.5 we have that $U - P(f)$ and $V - P(g)$ are domains. Restricting f and g to $U - P(f)$ and $V - P(g)$, respectively, we obtain holomorphic functions. Lemma 13.8 patches these together to give us a holomorphic function $h\colon W - (P(f) \cup P(g)) \to \mathbb{C}$. Since h agrees with

f in neighbourhoods of points of $P(f)$ (and similarly swapping f and g) we find that h has poles at exactly the points of $P(f) \cup P(g)$. So taking $P(h) = P(f) \cup P(g)$ completes the proof. □

We now have a companion result to Lemma 13.11.

Lemma 16.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Suppose that $z_0 \in P(h)$ is a pole of h. Let $N = \textsc{Ord}(h, z_0)$. Then there is a meromorphic function $g\colon U \to \hat{\mathbb{C}}$ so that $g(z_0) \neq 0, \infty$ and so that*

$$h(z) = (z - z_0)^N g(z)$$

Proof. Note that $N < 0$. We apply the removable singularity theorem (15.2) to

$$(z - z_0)^{-N+1} h(z)$$

This gives us a holomorphic function on $U - (P(h) - \{z_0\})$ vanishing at z_0. We factor this (using Lemma 13.11) to obtain g. All orders of poles of h and g agree away from z_0. □

Corollary 16.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Then the set of zeros $Z(h)$ is isolated in U.*

Proof. Since $h|(U - P(h))$ is holomorphic, the set $Z(h)$ is isolated in $U - P(h)$ (Lemma 14.7). By Lemma 16.10 we can factor h near a pole z_0 as $(z - z_0)^k$ times a non-vanishing holomorphic function. Thus the zeros of h do not accumulate at poles of h. Thus $Z(h)$ is isolated in U. □

Since poles are isolated we can "separate them" and apply Cauchy's theorem.

Corollary 16.12. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Suppose that $z_0 \in U$ is a pole of h. Then there is an $\epsilon > 0$ so that*

- *the circle $C = C(z_0; \epsilon)$ is disjoint from $P(h)$,*
- *the only pole of h in $B = B(z_0; \epsilon)$ is z_0,*
- *h has a Laurent expansion in B, and*
- *$\int_C h\,dz = 2\pi i \cdot \textsc{Res}(h, z_0)$.*

Proof. The existence of B and the Laurent expansion follows from Lemma 16.10. Suppose that g is the principal part of h in B. Let $f = h - g$ in B. So f is holomorphic. Note that g and h have the same residue at z_0, while f has none. We compute as follows:

$$\begin{aligned}
\int_C h\,dz &= \int_C (g+f)\,dz && \text{definition of } g \text{ and } f \\
&= \int_C g\,dz + \int_C f\,dz && \text{Lemma 2.13} \\
&= \int_C g\,dz && \text{Theorem 12.11} \\
&= 2\pi i \cdot \mathrm{Res}(g, z_0) && \text{Exercise 2.16} \\
&= 2\pi i \cdot \mathrm{Res}(h, z_0) && \text{definition of } g
\end{aligned}$$

This completes the proof. □

Exercise 16.13. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $g, f\colon U \to \hat{\mathbb{C}}$ are meromorphic, and are not identically zero in U. Prove, for all $z_0 \in U$, that $\mathrm{Ord}(fg, z_0) = \mathrm{Ord}(f, z_0) + \mathrm{Ord}(g, z_0)$ ◇

17. The residue theorem

The previous section gives us a version of contour integration for meromorphic functions. We must always arrange matters so that our contours avoid poles. This done, we obtain the *residue theorem*.

Theorem 17.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $C \subset U$ is a region as in Lemma 10.5. Suppose that $h\colon U \to \hat{\mathbb{C}}$ is meromorphic. Suppose that no poles of h lie on ∂C. Then we have*

$$\int_{\partial C} h\,dz = 2\pi i \cdot \sum \mathrm{Res}(h, w)$$

Here the sum is taken over $w \in P(h) \cap C$: poles of h lying in C.

Proof. Recall that $P(h)$ is isolated in U by definition (16.2). Since C is compact we have that $P(h) \cap C$ is finite (Exercise 14.3). We index these poles, taking $P(h) \cap C = (z_k)$. By finiteness, and by Corollary 16.12, we pick radii $r_k > 0$ to jointly satisfy the following.

- $B_k = B(z_k, r_k)$ is contained in the interior of C.
- h has a Laurent expansion in B_k.
- If $k \neq \ell$ then $\overline{B}_k \cap \overline{B}_\ell = \varnothing$.

We note that the C' again satisfies the hypotheses of Lemma 10.5. Also, h is holomorphic on an open neighbourhood of C'. Applying Theorem 9.2 we find that $\int_{\partial C'} h\,dz = 0$. From our orientation convention (Remark 10.6) and from Lemma 2.14 we deduce the following.

$$\int_{\partial C} h\,dz = \sum_k \int_{\partial B_k} h\,dz$$

Applying Corollary 16.12 we find that the right-hand side equals

$$2\pi i \cdot \sum_k \mathrm{Res}(h, z_k)$$

as desired. □

18. Definite integrals

I tried very hard not to spend time on your integrals, but to me the challenge of a definite integral is irresistible.

– G. H. Hardy
To H. M. S. Coxeter, letter of November 1926

The technique can be learned at the hand of typical examples, but even complete mastery does not guarantee success.

– Lars V. Ahlfors
Complex analysis: An introduction to the theory of analytic functions of one complex variable

18.1. **Terminology.** Recall that an integral of a real-valued function is *indefinite* if one or both bounds of integration are missing. The integral is *definite* if both bounds are supplied. A definite integral is *proper* if both bounds are finite, and the integrand is bounded (uniformly) in the domain of integration. It is *improper* if one of the bounds is plus (or minus) infinity, or if the integrand has a singularity of some type in the domain of integration.

There is a "standard procedure" when dealing with definite integrals. If improper, first check convergence. Next, evaluate. Finally, do some sanity checks, possibly including numerical ones.

Example 18.2. We consider the following improper definite integral.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

The integrand is even, and has no singularities in the domain. Note also that the integrand is less than $1/x^2$. So, to check convergence it suffices to show that $\int_1^\infty dx/x^2$ converges. We compute directly from the definition:

$$\begin{aligned} \int_1^\infty \frac{dx}{x^2} &= \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} \\ &= \lim_{R\to\infty} -1/x \,\big]_1^R \\ &= \lim_{R\to\infty} (-1/R) + 1 = 1 \end{aligned}$$

We now evaluate, using a substitution. We take $x = \tan(\theta)$ and find $dx = \frac{d\theta}{\cos^2(\theta)}$. In the range of interest, we have the "equalities"

$\tan(-\pi/2) = -\infty$ and $\tan(\pi/2) = \infty$. The change of variables gives the following.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

We double check our work, numerically, using SageMath [10].

```
sage: f = 1/(1 + x^2)
sage: numerical_integral(f, -Infinity, Infinity)
(3.1415926535897936, 5.155582396545663e-10)
```

The first entry is an estimate of the integral; the second is an estimate of the error. ◇

18.3. **Applying the residue theorem to Example 18.2.** Define $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ by $f(z) = 1/(1+z^2)$. This is meromorphic by Lemma 16.6. Factoring the denominator gives $(z-i)(z+i)$. Applying Lemma 16.6 again, we have that the poles of f are at $\pm i$ and are both simple. We compute the residue of f at i as follows.

$$\operatorname{Res}(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$$

We now build a suitable contour. Fix a real number $R > 1$. Let $B(R) = [-R, R]$ be the interval in $\mathbb{R} \subset \mathbb{C}$, oriented from $-R$ to R. Let $C(R)$ be the arc of the circle $C(0; R)$ in $\mathbb{C}$, oriented anticlockwise from R to $-R$. Let $A(R) = B(R) \cup C(R)$; this is a *half-circle contour*. By the residue theorem (17.1) the integral of f about the contour $A(R)$ is $2\pi i/2i = \pi$.

The integral of f along $B(R)$ tends to I as R tends to infinity. So, to prove $I = \pi$ it suffices to prove that $\int_{C(R)} f\,dz$ tends to zero. We apply the ML–inequality (2.22).

$$\begin{aligned}
\left|\int_{C(R)} f\,dz\right| &= \left|\int_0^{\pi} \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta\right| && \text{parametrising}\\
&= R\int_0^{\pi} \frac{d\theta}{|R^2e^{2i\theta}+1|} && \text{algebra}\\
&= R\int_0^{\pi} \frac{d\theta}{R^2-1} && \text{triangle inequality}\\
&= \pi\frac{R}{R^2-1} && \text{arclength}
\end{aligned}$$

As desired, this tends to zero as R tends to infinity.

18.4. **Another simple pole.** Here is another, similar, example. Fix $n \geq 2$ an integer. We evaluate the improper definite integral

$$I = \int_0^{\infty} \frac{dx}{1+x^n}$$

Define $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ by $f(z) = 1/(1+z^n)$. This is meromorphic by Lemma 16.6. Let $\omega = \textsc{Exp}(i\pi/n)$; this is a primitive $2n^{\text{th}}$ root of unity. Factoring the denominator of f, and applying Lemma 16.6, tells us that the poles of f lie at the odd powers of ω. Also, all poles are simple. The definition of the complex derivative gives us

$$\textsc{Res}(f,\omega) = \lim_{z\to\omega} \frac{z-\omega}{z^n+1} = \frac{1}{n\omega^{n-1}} = \frac{\omega^{n+1}}{n} = -\frac{\omega}{n}$$

We now build a suitable contour. Fix a real number $R > 2$. Let $B(R) = [0, R]$ be the interval in $\mathbb{R} \subset \mathbb{C}$, oriented from zero to R. Let $C(R)$ be the arc of the circle $C(0; R)$ in $\mathbb{C}$ running anticlockwise from R to $R\omega^2$. Let $D(R)$ be the line segment from $R\omega^2$ to zero in $\mathbb{C}$, oriented from $R\omega^2$ to zero. Let $A(R) = B(R) \cup C(R) \cup D(R)$; this is a *wedge contour* with angle $2\pi/n$. By the residue theorem (17.1) the integral of f about the contour $A(R)$ is

$$-2\pi i \cdot \frac{\omega}{n}$$

The integral of f along $B(R)$ tends to I as R tends to infinity. We parametrise $D(R)$ by $t \mapsto t\omega^2$. So the integral of f along $D(R)$ is as follows.

$$\int_{D(R)} \frac{dz}{z^n+1} = \int_R^0 \frac{\omega^2}{t^n+1}\,dt = -\omega^2 \int_0^R \frac{dt}{t^n+1}$$

This tends to $-\omega^2 I$ as R tends to infinity. So the integral of f along $D(R) \cup B(R)$ tends to $I - \omega^2 I = (1-\omega^2)I$.

Suppose that the integral along $C(R)$ tends to zero. From this we deduce that $(1-\omega^2)I = -2\pi i\omega/n$. Thus

$$I = \frac{-2\pi i\omega}{n(1-\omega^2)} = \frac{2\pi i}{n(\omega+\omega^{n-1})} = \frac{\pi/n}{\sin(\pi/n)}$$

It remains to prove that $\int_{C(R)} f\,dz$ tends to zero. We apply the ML–inequality (2.22).

$$\begin{aligned}
\left|\int_{C(R)} f\,dz\right| &= \left|\int_0^{\pi/n} \frac{iRe^{i\theta}}{1+R^n e^{ni\theta}}\,d\theta\right| && \text{parametrising}\\
&= R\int_0^{\pi/n} \frac{d\theta}{|1+R^n e^{ni\theta}|} && \text{algebra}\\
&= R\int_0^{\pi/n} \frac{d\theta}{R^n-1} && \text{triangle inequality}\\
&= \frac{\pi}{n}\frac{R}{R^n-1} && \text{arclength}
\end{aligned}$$

This tends to zero as R tends to infinity, as desired.

Here are two ways to check our work. If $n = 2$ then $(\pi/2)/\sin(\pi/2) = \pi/2$. This is half of the integral from negative to positive infinity (as computed above). We also take $n = 100$ and compute numerically.

```
sage: n = 100
sage: f = 1/(1 + x^n)
sage: numerical_integral(f, 0, Infinity)
(1.0001645122969123, 8.052568059024878e-07)
sage: ((pi/n)/sin(pi/n)).n()
1.00016451234931
```

18.5. **A trigonometric integral.** We first evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2}\,dx$$

Since $\sin(x)$ is bounded the integral converges. Since $\sin(x)$ is odd the integral vanishes.

The following integral is more difficult.

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx$$

It is again convergent and for the same reason. Define $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ by $f(z) = \textsc{Exp}(iz)/(1+x^2)$. This is meromorphic by Lemma 16.6. Since $\textsc{Exp}$ is non-vanishing, we have the following:

$$\textsc{Res}(f,i) = \lim_{z\to i}(z-i)f(z) = \textsc{Exp}(-1)\cdot \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2ie}$$

We again use the half-circle contour. Fix $R > 1$. Set $B(R) = [-R, R]$, set $C(R)$ to be the upper half of $C(0;R)$, and take $A(R) = B(R) \cup C(R)$. By the residue theorem (17.1) the integral of f about $A(R)$ is $2\pi i/2ie = \pi/e$.

We now claim that the integral of f along $C(R)$ tends to zero. Along $C(R)$ we have $\textsc{Imag}(z) \geq 0$; this implies that $|\,\textsc{Exp}(iz)| \leq 1$. So the ML–inequality (2.22) applies as before, proving the claim.

Finally, the imaginary part of $\int_{B(R)} f\,dz$ vanishes. Thus the real part tends to π/e. So this is the limit of the improper integral.

As usual, we check our work numerically. The oscillations of cosine near infinity reduce the accuracy of the default numerical integrator in SageMath. We can reduce the error a bit by asking the algorithm to consider more points.

```
sage: h = cos(x)/(1 + x^2)
sage: Inf = Infinity
sage: numerical_integral(h, -Inf, Inf, max_points = 10000)
(1.1557270795777321, 1.4110896973118514e-06)
sage: (pi/e).n()
1.15572734979092
```

19. Trivial first homology

19.1. **Homeomorphisms.** Suppose that X and Y are topological spaces. Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are continuous maps and are inverses (that is, both right and left inverses). Then we call f and g *homeomorphisms* and we say that X and Y are *homeomorphic*. Note that "being homeomorphic" is an equivalence relation.

Note that, if X and Y are homeomorphic, it is often the case that there are many homeomorphisms between them. For example, the set of self-homeomorphisms of the real line gives a large (and interesting) group.

Exercise 19.2. Give an explicit homeomorphism between the open unit disc $\mathbb{D} = B(0; 1)$ and the complex plane $\mathbb{C}$. ◇

Exercise 19.3. Suppose that $0 < a < b$ is a pair of real numbers. Recall that $A(0; a, b)$ is an open annulus in $\mathbb{C}$ (10.3). Suppose that $0 < c < d$ is another pair of real numbers. Prove that $A = A(0; a, b)$ is homeomorphic to $C = A(0; c, d)$. ◇

Definition 19.4. Fix an angle $\theta \in \mathbb{R}$. Define the *ray* $R(\theta)$ to be

$$R(\theta) = \{z \in \mathbb{C}^\times \mid \theta \in \textsc{Arg}(z)\} \cup \{0\}$$

and the *cut plane* $C(\theta)$ to be

$$C(\theta) = \mathbb{C} - R(\theta)$$

In this setting, we may also call $R(\theta)$ a *branch cut*. ◇

Note that $C(\theta)$ is homeomorphic to $C(\eta)$ for any other angle η; simply rotate $C(\theta)$ through an angle of $\eta - \theta$.

Exercise 19.5. Give an explicit homeomorphism between the *right half plane*

$$U = \{z \in \mathbb{C} \mid \textsc{Real}(z) > 0\}$$

and the cut plane $C(\pi)$. ◇

More abstract spaces may also be homeomorphic to domains.

Exercise 19.6. Prove that the annulus $A = A(0; 1, 2)$ is homeomorphic to the *cylinder* $C = S^1 \times (0, 1)$. ◇

19.7. **Invariance.** Suppose that $U \subset \mathbb{C}$ is a domain. We say that U has *trivial first homology* if $H_1(U) = 0$. That is, the first homology group of U vanishes. Unwrapping the definitions a bit, this happens exactly when every one-cycle in U is a one-boundary in U.

Example 19.8. In Exercise 9.3 we found a one-cycle in $\mathbb{C}^\times$ which is not a one-boundary in $\mathbb{C}^\times$. We deduce that $H_1(\mathbb{C}^\times) \neq 0$. The same argument applies to the punctured disc $\mathbb{D}^\times$ and to the annulus $A(0; a, b)$. ◇

In fact, $H_1(\mathbb{C}^\times) \cong \mathbb{Z}$. Also, domains with "more holes" have more homology. We will not pursue these lines of inquiry.

Exercise 19.9. Suppose that $U \subset \mathbb{C}$ is a convex domain. Prove that $H_1(U) = 0$. ◇

Exercise 19.10. Suppose that U and V are domains in $\mathbb{C}$. Suppose that U is homeomorphic to V. Then we have $H_1(U) = 0$ if and only if $H_1(V) = 0$. ◇

Exercise 19.11. Prove that the plane $\mathbb{C}$ is not homeomorphic to the punctured plane $\mathbb{C}^\times$ (or the punctured disc, or an annulus). ◇

More generally, suppose that U and V are homeomorphic domains. Then any homeomorphism $f\colon U \to V$ induces an isomorphism

$$f_*\colon H_1(U) \to H_1(V)$$

We will not pursue this line of inquiry.

20. The logarithm

It's Log, it's Log,
It's big, it's heavy, it's wood.
It's Log, it's Log,
It's better than bad, it's good!
– The Ren & Stimpy Show
Log, from Blammo!

20.1. Branches of the logarithm.

Lemma 20.2. *Suppose that $U \subset C$ is a domain. Suppose that U has trivial first homology. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f has a primitive in U.*

Exercise 20.3. Prove Lemma 20.2. ◇

Theorem 20.4. *Suppose that $U \subset C$ is a domain. Suppose that U has trivial first homology. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $f(z) \neq 0$ for all $z \in U$. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $f = \textsc{Exp} \circ g$.*

Note that $g(z_0)$ lies in $\textsc{Log}(f(z_0))$ for all $z_0 \in U$. That is, g is a branch of $\textsc{Log} \circ f$ (Definition 2.6). Branches are typically not unique. However, suppose that $h\colon U \to \mathbb{C}$ is another branch of $\textsc{Log} \circ f$. Then the difference $g - h$ is holomorphic and in $2\pi i\mathbb{Z}$, so is constant.

Proof of Theorem 20.4. By Theorem 12.11 the derivative f' is holomorphic in U. Since f is never zero, by Example 4.5 the ratio f'/f is holomorphic in U. Since U has trivial first homology, by Lemma 20.2 the ratio f'/f has a primitive, say G, in U. Fix any z_0 in U. Note that $(\textsc{Exp} \circ G)(z_0)/f(z_0) = k$ is not zero. Fix any $K \in \textsc{Log}(k)$; define $g = G - K$. So g is again a primitive for f'/f. We now define $h = (\textsc{Exp} \circ g)/f$. So h is holomorphic by Examples 4.6 and 4.5.

We compute:

$$h(z_0) = \frac{\textsc{Exp}(g(z_0))}{f(z_0)} = \frac{\textsc{Exp}(G(z_0) - K)}{f(z_0)} = \frac{\textsc{Exp}(G(z_0))}{f(z_0)} \cdot \frac{1}{k} = 1$$

Also

$$h' = \frac{(\textsc{Exp} \circ g) \cdot g' \cdot f - (\textsc{Exp} \circ g) \cdot f'}{f^2} = (\textsc{Exp} \circ g) \cdot \frac{g' \cdot f - f'}{f^2} = 0$$

So h is constant and thus identically equal to one. So $\textsc{Exp} \circ g = f$, as desired. □

As a very special case, we take U to be a cut plane and $f(z) = z$. Theorem 20.4 tells us to integrate dz/z, the element of the logarithm, to find a branch g of $\textsc{Log}(z)$. We deduce that g is holomorphic and satisfies $\textsc{Exp}(g(z)) = z$ for all z in U. That is, g is a holomorphic branch of $\textsc{Log}$ on the domain U.

Corollary 20.5. *With hypotheses as in Theorem 20.4, suppose that $N > 0$ is a natural number. Then there is a holomorphic function $g \colon U \to \mathbb{C}$ so that $f = g^N$.*

Proof. Let $h \colon U \to \mathbb{C}$ be a branch of the logarithm of f. Define $g \colon U \to \mathbb{C}$ by $g(z) = \textsc{Exp}(h(z)/N)$. Note that g is a composition of holomorphic functions, so is holomorphic. Also, $f = g^N$ by the addition law for $\textsc{Exp}$ and the definition of h. □

Using this we can prove an important "change of coordinates" principle.

Proposition 20.6. *Suppose that $U \subset C$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that f is non-constant. Fix $z_0 \in U$ and set $N = \textsc{Ord}(f - f(z_0), z_0)$. Then there is an $\epsilon > 0$ and a holomorphic function $F \colon B(z_0; \epsilon) \to \mathbb{C}$ so that $F'(z_0) \neq 0$ and $f(z) - f(z_0) = (F(z))^N$.*

Proof. By Lemma 13.4 we have that $N < \infty$. By Lemma 13.11 we have some $g \colon U \to \mathbb{C}$ which is holomorphic, which is non-zero at z_0, and which has $f(z) - f(z_0) = (z - z_0)^N g(z)$. Since g is continuous, there is some $\epsilon > 0$ so that g has no roots in $B = B(z_0; \epsilon)$. By Corollary 20.5 we have some $h \colon B \to \mathbb{C}$ which is holomorphic and which has $g = h^N$. We deduce that h is not zero at z_0.

Taking powers we find $f(z) - f(z_0) = [(z - z_0)h(z)]^N$. Define $F(z) = (z - z_0)h(z)$. Note that $F(z_0) = 0$, that F is holomorphic in B, and that $F'(z_0) = h(z_0) \neq 0$. Finally, $f(z) - f(z_0) = [F(z)]^N$, as desired. □

20.7. **A logarithmic integral.**

Branch points are mathematics;
branch cuts are sociology.
– Curt McMullen
Advanced Complex Analysis

We first evaluate the improper integral

$$I = \int_0^\infty \frac{\ln(x)}{1+x^2}\,dx$$

For large x we have $\ln(x) < x^{1/4}$. Comparing the integrand with $1/x^{7/4}$ proves convergence of

$$G = \int_1^\infty \frac{\ln(x)}{1+x^2}\,dx$$

We now consider the change of variables $x = 1/u$. So $dx = -du/u^2$. Thus the integral immediately above becomes

$$G = \int_1^0 \frac{\ln(1/u)}{1+1/u^2}\frac{-du}{u^2} = -\int_0^1 \frac{\ln(u)}{u^2+1}\,du$$

Thus I also converges near zero. And finally, $I = -G + G = 0$.

Here is the next integral.

$$J = \int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx$$

Here we use $\ln^2(x)$ to mean $(\ln(x))^2$. Note that the integrand is positive on the domain of integration; so the integral does not vanish. Convergence near infinity is proven using $\ln^2(x) < x^{1/2}$. Convergence near zero is again proven using the substitution $x = 1/u$.

We now build a holomorphic function. Let $U = C(3\pi/2)$ be the cut plane where we have removed the ray (including zero) along the negative imaginary axis. Let g be the branch of $\textsc{Log}(z)$ in U chosen so that $g(1) = 0$ (Theorem 20.4). So, for positive real x we have

$$g(x) = \ln(x) \qquad \text{and} \qquad g(-x) = \ln(x) + \pi i$$

Note also that $g(i) = g(e^{\pi i/2}) = \pi i/2$. We use $g^2(z)$ to denote $(g(z))^2$.

As g is holomorphic we may apply Example 4.5 to find that $h(z) = g^2(z)/(z^2+1)$ is meromorphic in U. This has only one pole at $z = i$ and this pole is simple. Accordingly we have the following.

$$\textsc{Res}(h, i) = \frac{g^2(i)}{2i} = \frac{g^2(e^{\pi i/2})}{2i} = \frac{-\pi^2/4}{2i} = \frac{\pi^2 i}{8}$$

We now build a suitable contour. Fix real numbers $0 < \epsilon < 1 < R$. Let $B(\epsilon, R) = [\epsilon, R]$ and $D(R, \epsilon) = [-R, -\epsilon]$ be the given intervals in $\mathbb{R} \subset \mathbb{C}$. Let $C(\epsilon)$ and $C(R)$ be the arcs of the circles $C(0;\epsilon)$ and

$C(0;R)$, respectively, lying in the upper-half plane. We orient $C(R)$ anti-clockwise and $C(\epsilon)$ clockwise. Define

$$A(\epsilon,R) = B(\epsilon,R) \cup C(R) \cup D(R,\epsilon) \cup C(\epsilon)$$

This is (half of) a *keyhole contour*. By the residue theorem (17.1) the integral of h about $A(\epsilon,R)$ is

$$2\pi i \cdot \frac{\pi^2}{8} i = -\frac{\pi^3}{4}$$

The integral of h along $B(\epsilon,R)$ tends to J (our desired answer) as ϵ tends to zero and R tends to infinity. The integral along $D(R,\epsilon)$ is similar; we compute as follows.

$$\begin{aligned}
\int_{D(R,\epsilon)} \frac{g^2(z)}{1+z^2}\,dz &= \int_{-R}^{-\epsilon} \frac{g^2(x)}{1+x^2}\,dx \\
&= \int_{R}^{\epsilon} \frac{g^2(-y)}{1+(-y)^2}\,d(-y) \\
&= \int_{\epsilon}^{R} \frac{g^2(-y)}{1+y^2}\,dy \\
&= \int_{\epsilon}^{R} \frac{(\ln(y)+\pi i)^2}{1+y^2}\,dy \\
&= \int_{\epsilon}^{R} \frac{\ln^2(y)+2\pi i\ln(y)-\pi^2}{1+y^2}\,dy
\end{aligned}$$

This last tends to

$$J + 2\pi i \cdot 0 - \pi^2 \cdot \frac{\pi}{2} = J - \frac{\pi^3}{2}$$

as ϵ tends to zero and R tends to infinity.

Exercise 20.8. The integral of h along $C(\epsilon)$ tends to zero as ϵ tends to zero; The integral of h along $C(R)$ tends to zero as R tends to infinity. ◇

So the residue theorem implies

$$-\frac{\pi^3}{4} = 2J - \frac{\pi^3}{2} \qquad \text{and so} \qquad J = \frac{\pi^3}{8}$$

We finally check our work numerically.

```
sage: h = (ln(x))^2/(1 + x^2)
sage: numerical_integral(h, 0, Infinity)
(3.87578458209479, 2.9038350124466206e-06)
sage: (pi^3/8).n()
3.87578458503748
```

21. Entire functions

A holomorphic function is *entire* if its domain is the complex plane. Here is a rephrasing of Corollary 12.9.

Lemma 21.1. *Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is entire. Then the series expansion of h about zero has infinite radius of convergence.* □

Theorem 21.2. *Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then h is constant.*

This is usually called *Liouville's theorem*; for a history, see [5, pages 534-548]. Cauchy's theory gives a short proof as follows.

Proof of Theorem 21.2. Suppose that $A = (a_n)$ are the coefficients of the series expansion of h, as given by Theorem 12.11. Since h is bounded, there is some $M > 0$ so that $|h(z)| \leq M$ for all $z \in \mathbb{C}$. We now fix any $r > 0$ and apply the estimate of Lemma 12.5. We deduce, for all $n \geq 0$, that $|a_n| \leq M/r^n$. When $n = 0$ this only tells us that $|h(0)| = |a_0| \leq M$, which we already know. When $n > 0$ letting r tend to infinity tells us that $a_n = 0$. Thus $h = h(0)$ is constant. □

One application (due to Liouville [5, page 544]) is a very short proof of the fundamental theorem of algebra.

Theorem 21.3. *Suppose that $p \in \mathbb{C}[z]$ is a non-constant polynomial. Then p has a complex root.*

Proof. We prove the contrapositive. So, suppose that p is a complex polynomial without roots. In particular, p is not identically zero in $\mathbb{C}$. Note that p is holomorphic in $\mathbb{C}$ by Example 4.3. Since p has no root, $1/p$ is holomorphic by Example 4.5. Thus $1/p$ is entire. Since $|1/p|$ is bounded near infinity, it is bounded. By Theorem 21.2 we deduce that $1/p$, and thus p, is constant. □

Exercise 21.4. Suppose that $p \in \mathbb{C}[z]$ is not the zero polynomial. Prove that $|1/p|$ is bounded near infinity. ◇

22. The maximum modulus principle

We prove the maximum modulus principle and several of its consequences.

22.1. The maximum modulus principle.

Theorem 22.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $|f|$ has a local maximum. Then f is constant.*

Proof. Suppose that $z_0 \in U$ is a local maximum of $|f|$. Set $M_0 = |f(z_0)|$. Since U is open and since z_0 is a local maximum we have some $r > 0$ so that:

- $\overline{B(z_0; r)} \subset U$ and
- $|f(z)| \leq M_0$ for all z in $\overline{B(z_0; r)}$.

The mean value theorem (11.3) gives:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Taking the modulus of both sides we find:

$$M_0 = |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq M_0$$

We deduce that the integral of $M_0 - |f(z_0 + re^{i\theta})|$ is zero. Thus $|f(z)| = M_0$ for all $z \in \partial B(z_0; r)$.

Suppose that $M_0 = 0$. Then the identity theorem (14.10) implies that f vanishes identically in U. In this case we are done.

Suppose instead that $M_0 \neq 0$. Set $g(z) = f(z)/f(z_0)$. So $g(z_0) = 1$. Also, $|g(z)| = 1$ for all $z \in \partial B(z_0; r)$. To simplify the notation we define $C, S \colon [0, 2\pi] \to \mathbb{R}$ by:

$$C(\theta) = \text{Real}(g(z_0 + re^{i\theta}))$$
$$S(\theta) = \text{Imag}(g(z_0 + re^{i\theta}))$$

So C and S are continuous and satisfy $C^2(\theta) + S^2(\theta) = 1$. The mean value theorem now gives:

$$1 = \frac{1}{2\pi} \int_0^{2\pi} (C(\theta) + iS(\theta))\, d\theta$$

Taking the real part we find that the integral of $1 - C(\theta)$ is zero. Thus $C(\theta)$ is constant and equal to one. So $S(\theta)$ is constant and equal to zero. The identity theorem (14.10) now implies that g, and thus f, is constant. □

Here is another common version of the maximum modulus principle.

Corollary 22.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $R \subset U$ is a region as in Lemma 10.5. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not constant. Then the maximum of $|f|$, on R, is not realised in the interior of R.* □

22.4. **The minimum modulus principle.** We give a version of the minimum modulus principle suitable for detecting zeros.

Theorem 22.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that $|f|$ has a local minimum at $z_0 \in U$. Suppose that $f(z_0) \neq 0$. Then f is constant.*

Proof. Set $m_0 = |f(z_0)|$. Since U is open, since z_0 is a local minimum, and since f is continuous we have some $r > 0$ so that:

- $B(z_0; r) \subset U$,
- $m_0 \leq |f(z)|$ for all $z \in B(z_0; r)$, and

- $f(z) \neq 0$ for $z \in B(z_0; r)$.

Define $g\colon B(z_0, r) \to \mathbb{C}$ by $g(z) = 1/f(z)$. So g is holomorphic. Thus $M_0 = 1/m_0$ is a local maximum for $|g|$ in V. So, by the maximum modulus principle (22.2), we have that g is constant. So, by the identity theorem (14.10), we have that f is constant. □

22.6. **The open mapping theorem.**

Theorem 22.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is non-constant. Then $f(U)$ is open.*

Proof of Theorem 22.7. Suppose that z_0 lies in U. Take $w_0 = f(z_0)$. Define $g\colon U \to \mathbb{C}$ by $g(z) = f(z) - w_0$. Note that g is holomorphic, is non-constant, and has a zero at z_0.

Recall that U is open and that the zeros of g are isolated (Lemma 14.7). So we can pick $r > 0$ so that:

- $\overline{B(z_0; r)} \subset U$ and
- $\overline{B(z_0; r)} \cap Z(g) = \{z_0\}$.

Define

$$2\epsilon = \min\{|g(z)|\colon z \in C(z_0; r)\}$$

Claim. $B(w_0; \epsilon) \subset f(B(z_0; r))$.

Proof. Fix $w \in B(w_0; \epsilon)$. Define $g_w\colon U \to \mathbb{C}$ by $g_w(z) = f(z) - w$; note that g_w is holomorphic and is non-constant. So we have:

$$|g_w(z_0)| = |f(z_0) - w| = |w_0 - w| < \epsilon$$

Suppose that z is any point of $C(z_0; r)$. We compute as follows:

$$\begin{aligned} |g_w(z)| &= |f(z) - w| \\ &= |f(z) - w_0 + w_0 - w| \\ &\geq |f(z) - w_0| - |w_0 - w| \\ &\geq |g(z)| - |w_0 - w| \\ &> 2\epsilon - \epsilon = \epsilon \\ &> |g_w(z_0)| \end{aligned}$$

We deduce that $|g_w|$ has a local minimum in $B(z_0; r)$, say at z_1. Since g_w is non-constant, we may apply the (contrapositive of the) minimum modulus principle (Theorem 22.5) to $g_w|B(z_0; r)$. This implies that $g_w(z_1) = 0$. So $f(z_1) = w$. Thus w lies in $f(B(z_0; r))$, as desired. □

By the claim we have $B(w_0; \epsilon) \subset f(U)$. Thus $f(U)$ is open, as desired. □

Here is a useful corollary of the open mapping theorem.

Exercise 22.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and injective. Set $V = f(U)$. Prove that V is a domain. ◇

23. The plane

The rest of the notes deals with our final topic: the classification of domains up to *conformal equivalence*.

23.1. **Conformal equivalence.**

Definition 23.2. Suppose that U and V are domains in $\mathbb{C}$. Suppose that $f\colon U \to V$ is holomorphic. Suppose that $g\colon V \to U$ is holomorphic. Suppose that $f \circ g = \mathrm{Id}_V$ and $g \circ f = \mathrm{Id}_U$. Then we say that f is a *biholomorphic*. ◇

We may also call f a *conformal equivalence*; we may also say that U and V are *conformally equivalent*. The open mapping theorem gives us many conformal equivalences.

Proposition 23.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and injective. Then $f'(z) \neq 0$ to all $z \in U$. Futhermore, setting $V = f(U)$, we have that f is a conformal equivalence from U to V.*

Proof. Fix any z_0 in U. Suppose, to obtain a contradiction, that $f'(z_0) = 0$. If $f - f(z_0)$ vanishes to infinite order then, by Lemma 13.4, we have that $f - f(z_0)$ vanishes identically in U. In this case f is constant, so not injective, a contradiction. Suppose instead that $N = \mathrm{ORD}(f - f(z_0), z_0) < \infty$. Since $f - f(z_0)$ vanishes at z_0 we deduce that $N \geq 2$. By Proposition 20.6 there is a small neighbourhood B of z_0 and a holomorphic mapping $F\colon B \to \mathbb{C}$ so that $F'(z_0) \neq 0$ and $f(z) - f(z_0) = (F(z))^N$. So $F(z_0) = 0$. We apply the open mapping theorem (Theorem 22.7) to find that $F(B)$ contains a small neighbourhood C of zero. Since $N \geq 2$, the function $z \mapsto z^N$ is not injective on C. Thus $f(z) - f(z_0) = (F(z))^N$ is not injective on B. So $f(z) = f(z_0) + (F(z))^N$ is not injective on U. This is the desired contradiction.

We now produce a holomorphic inverse for f. Recall that $V = f(U)$ is a domain (Exercise 22.8). Define $g\colon V \to U$ by $g(w) = z$ iff $f(z) = w$. This is well-defined because $f\colon U \to V$ is a bijection. Applying the open mapping theorem, we have that g is continuous.

It remains to show that g is holomorphic. Suppose that w and w_0 are distinct points in V. Since $V = f(U)$ there are points z and z_0 in U so that $f(z) = w$ and $f(z_0) = w_0$. Note that:

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

Since g is continuous, as w tends to w_0, the point z tends to z_0. Since $f'(z_0)$ is not zero, the limit of the right-hand side is $1/f'(z_0)$. We deduce that $g'(w_0) = 1/f'(z_0)$. Thus g is holomorphic. □

Recall that holomorphic functions are continuous (4.7). Thus biholomorphisms are homeomorphisms. So, non-homeomorphic domains are never conformally equivalent. As an example, in Exercise 19.11 we proved that the plane $\mathbb{C}$ and the punctured plane $\mathbb{C}^\times$ are not homeomorphic. Thus $\mathbb{C}$ and $\mathbb{C}^\times$ are not conformally equivalent.

On the other hand, in Exercise 19.2 we proved that the disc $\mathbb{D}$ *is* homeomorphic to $\mathbb{C}$. However, conformally equivalence is a finer relation than homeomorphism:

Lemma 23.4. *The complex plane* $\mathbb{C}$ *is not conformally equivalent to the unit disc* $\mathbb{D}$*.*

Proof. Suppose that $f\colon \mathbb{C} \to \mathbb{D}$ is holomorphic. So it is entire and bounded. So it is constant, by Liouville's theorem (21.2). □

Here is a generalisation.

Lemma 23.5. *Suppose that* P *and* Q *are isolated subsets of* $\mathbb{C}$ *and* $\mathbb{D}$*, respectively. Then* $\mathbb{C} - P$ *is not conformally equivalent to* $\mathbb{D} - Q$*.*

Proof. Suppose that $f\colon \mathbb{C} - P \to \mathbb{D}$ is holomorphic. Note that f is bounded. So, by the removable singularity theorem (15.2), for each $p \in P$ there is a holomorphic extension of f to $f_p\colon \mathbb{C} - (P - \{p\}) \to \mathbb{D}$. We now patch together all of the maps f_p (Lemma 13.8) to obtain a holomorphic function $F\colon \mathbb{C} \to \mathbb{D}$. Liouville's theorem (21.2) implies that F is constant, so f is constant. □

23.6. **Automorphisms.** Suppose that $U \subset \mathbb{C}$ is a domain. We often call a biholomorphic function $f\colon U \to U$ a *(conformal) automorphism* of U. As usual, we suppress the adjective "conformal" when it is clear from context. We take $\textsc{Aut}(U)$ to be the group of biholomorphisms of U; the group multiplication is function composition.

Lemma 23.7. *Suppose that* $U \subset \mathbb{C}$ *is a domain. Suppose that* $f\colon U \to U$ *is biholomorphic. Suppose that* $z_0 \in U$*. Then* $f'(z_0) \neq 0$*.*

Proof. Suppose that $g\colon U \to U$ is the given biholomorphic inverse to f. So $g \circ f = \mathrm{Id}_U$. By the chain rule, we have $g'(f(z_0))f'(z_0) = 1$. Thus $f'(z_0) \neq 0$. □

23.8. **Automorphisms of the plane.** Recall (Exercise 1.8) that $\textsc{Sim}(\mathbb{C})$ is the group of *similarities* of $\mathbb{C}$.

Theorem 23.9. *All similarities of* $\mathbb{C}$ *are automorphisms; all automorphisms of* $\mathbb{C}$ *are similarities.*

That is, $\textsc{Aut}(\mathbb{C}) = \textsc{Sim}(\mathbb{C})$.

Proof of Theorem 23.9. Suppose that $f(z) = az + b$ is a similarity of $\mathbb{C}$. Note that $a \neq 0$. So $g(z) = (z - b)/a$ is the inverse similarity. So f is biholomorphic, and thus is an automorphism.

Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is a (biholomorphic) automorphism of $\mathbb{C}$. Let $\mathbb{D}$ be the open unit disc. Let $U = \mathbb{C} - \overline{\mathbb{D}}$. So $\mathbb{D}$ and U are disjoint domains. Since h is a homeomorphism, $h(\mathbb{D})$ and $h(U)$ are open and disjoint. In particular $h(U)$ is not dense in $\mathbb{C}$.

Note that h is entire. Thus h has a series expansion with infinite radius of convergence (Lemma 21.1). Let $A = (a_n)$ be the coefficients of this series. Recall that $h(U)$ is not dense in $\mathbb{C}$. Thus $h(1/z)$ does not have an essential singularity at zero (Theorem 15.13).

We deduce that A has only finitely many non-zero terms. So h is a polynomial. By the fundamental theorem of algebra (Theorem 21.3), the number of roots of h equals the degree of h. But h is a biholomorphic, so bijective, so has exactly one root. Thus h has degree one and so is a similarity, as desired. □

The action of $\textsc{Aut}(\mathbb{C}) = \textsc{Sim}(\mathbb{C})$ on $\mathbb{C}$ is *(uniquely) two-transitive*, as follows.

Exercise 23.10. Suppose that u and v are distinct points of $\mathbb{C}$. Then there is a unique similarity sending u and v (in that order) to 0 and 1 (in that order). ◇

24. The extended plane

In Section 16 we defined

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

to be the *extended plane*. We call ∞ the *point at infinity*. In this section we discuss the various structures on $\hat{\mathbb{C}}$.

24.1. The topology of the extended plane.

We equip the extended plane with a topology as follows. A subset $U \subset \hat{\mathbb{C}}$ is open if

- $\infty \notin U$ and U is open in $\mathbb{C}$ or
- $\infty \in U$ and $\hat{\mathbb{C}} - U$ is a compact subset of $\mathbb{C}$.

For example, the sets $(\mathbb{C} - \overline{B(0; R)}) \cup \{\infty\}$ form an open neighbourhood basis of the point at infinity. (The extended plane is also called the *one-point compactification* of $\mathbb{C}$.)

Exercise 24.2. Sketch a proof that $\hat{\mathbb{C}}$ is homeomorphic to S^2, the unit two-sphere in $\mathbb{R}^3$. In particular, $\hat{\mathbb{C}}$ is compact. ◇

Exercise 24.3. Suppose that $L \subset \mathbb{C}$ is a line. Suppose that $\hat{L}$ is the closure of L in $\hat{\mathbb{C}}$. What is the homeomorphism type of $\hat{L}$? ◇

Suppose that $w \in \mathbb{C}^\times$ is a complex number. We define two functions from $\hat{\mathbb{C}}$ to itself:

$$A_w(z) = \begin{cases} z + w, & \text{if } z \neq \infty \\ \infty, & \text{if } z = \infty \end{cases} \qquad M_w(z) = \begin{cases} zw, & \text{if } z \neq \infty \\ \infty, & \text{if } z = \infty \end{cases}$$

(We use A_0 to denote the identity function on $\hat{\mathbb{C}}$. We use M_0 and M_∞ to denote the constant functions with values zero and infinity, respectively. Note that we do not define the "product" of zero and infinity.)

We define *inversion* $V \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$V(z) = \begin{cases} 1/z & \text{if } z \neq 0, \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty \end{cases}$$

Exercise 24.4. Suppose that $w \in \mathbb{C}^\times$. Prove that A_w, M_w, and V are homeomorphisms of the extended plane. ◇

We often abuse notation by omitting the symbols for A, M, and V; instead we "overload" the usual notations of addition, multiplication, and reciprocation. However, note that on points of $\hat{\mathbb{C}}$ these are not arithmetic operations, but rather transformations of the extended plane.

Exercise 24.5. Understand how the inversion $V(z) = 1/z$ transforms circles and lines in the plane. ◇

24.6. **The conformal structure of the extended plane.** Away from infinity, the extended plane is just the plane, and so has its usual notion of signed angle. If two oriented arcs cross at infinity, then we apply the inversion V, measure their signed angle at the origin, and use that to define the original signed angle at infinity. Thus, inversion is conformal "by construction" at infinity.

Here is the main example of computing an angle at infinity. Suppose that L and M are distinct lines, through the origin, in $\mathbb{C}$. Orient L and M. Define $\hat{L} = L \cup \infty$ and $\hat{M} = M \cup \infty$. So the oriented loops $\hat{L}$ and $\hat{M}$ cross at zero and at infinity.

Suppose that, at the origin, the positive real axis makes angles λ and μ with the (now oriented) lines L and M, respectively. Suppose that $0 < \lambda < \mu < \pi$. So, anticlockwise, the oriented lines come in the following order:

- the positive reals,
- then L, and
- then M.

So the signed angle from L to M is $\mu - \lambda$. The images $V(\hat{L})$ and $V(\hat{M})$ are again oriented loops. Deleting the point at infinity gives $L' = V(\hat{L}) - \{\infty\}$ and $M' = V(\hat{M}) - \{\infty\}$. Note that L' and M' are again oriented lines (Exercise 24.5). They make signed angles of $\pi - \lambda$ and $\pi - \mu$ with the positive real axis. Following the instructions given by the orientations, we have $0 < \pi - \mu < \pi - \lambda < \pi$. In particular, anticlockwise, the oriented lines come in the following order:

- the positive reals,
- then M', and
- then L'.

So the signed angle from L' to M' is $2\pi - \mu + \lambda$.

Exercise 24.7. Suppose that L and M are disjoint parallel lines in $\mathbb{C}$. Let $\hat{L}$ and $\hat{M}$ be the closures of L and M in $\hat{\mathbb{C}}$. Sketch $L' = V(\hat{L}) - \{\infty\}$ and $M' = V(\hat{M}) - \{\infty\}$. [There are five cases.] ◇

24.8. **The automorphisms of the extended plane.** A function $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *bimeromorphic* if it is meromorphic with a meromorphic inverse. We also call such an f a *(conformal) automorphism* of $\hat{\mathbb{C}}$. (Compare to the definitions given in Section 23.6.) We take $\mathrm{A{\scriptstyle UT}}(\hat{\mathbb{C}})$ to be the group of bimeromorphisms of $\hat{\mathbb{C}}$; the group multiplication is function composition.

Lemma 24.9. *The functions A_w, M_w, and V (as defined in Section 24.1 and with $w \in \mathbb{C}^\times$) are automorphisms of $\hat{\mathbb{C}}$.*

Proof. As noted in Exercise 24.4 the functions have inverses A_{-w}, $M_{1/w}$, and V, respectively.

Note that
$$\frac{1}{A_w(1/z)} = \frac{1}{1/z + w} = \frac{z}{1 + zw}$$
Thus A_w is meromorphic at infinity.

Note that
$$\frac{1}{M_w(1/z)} = \frac{1}{w/z} = \frac{z}{w}$$
Thus M_w is meromorphic at infinity.

Finally, V is meromorphic at infinity by definition. □

Lemma 24.10. *Suppose that a, b, c, and d lie in $\mathbb{C}$, with $ad - bc \neq 0$. Then the rational map $f(z) = (az + b)/(cz + d)$ is an automorphism of $\hat{\mathbb{C}}$. Furthermore, all automorphisms of $\hat{\mathbb{C}}$ are of this form.*

Degree one rational maps are called *linear fractional transformations.* Another very common name for these is *Möbius transformations.*

Proof of Lemma 24.10. Suppose that $f(z) = (az + b)/(cz + d)$ is a linear fractional transformation. Then f is a composition of additions, multiplications, and inversions. We deduce from Lemma 24.9 that f is an automorphism of $\hat{\mathbb{C}}$.

Suppose instead that f is an automorphism. Suppose that f fixes ∞. So $f|\mathbb{C}$ is an automorphism of $\mathbb{C}$. Thus $f|\mathbb{C}$ is a similarity (Theorem 23.9).

Suppose that f does not fix ∞. Suppose that $f(\infty) = p$. Define $g(z) = \frac{1}{f(z)-p}$. So g is an automorphism of $\hat{\mathbb{C}}$ that fixes ∞. So $h = g|\mathbb{C}$ is a similarity. Suppose that $h(z) = cz + d$, with $c \in \mathbb{C}^\times$. So we have

$$\begin{aligned} cz + d &= \frac{1}{f(z) - p} && \text{thus} \\ f(z) - p &= \frac{1}{cz + d} && \text{and so} \\ f(z) &= p + \frac{1}{cz + d} \\ &= \frac{(pc)z + (pd + 1)}{cz + d} \end{aligned}$$

Noting that $pcd - (pd + 1)c = -c$ is non-zero, we are done. □

24.11. **The automorphisms of the punctured plane.**

Theorem 24.12. *Suppose that a lies in $\mathbb{C}^\times$. Then $f(z) = az$ and $g(z) = a/z$ are automorphisms of $\mathbb{C}^\times$. Furthermore, all automorphisms of $\mathbb{C}^\times$ have one of these two forms. So* $\textsc{Aut}(\mathbb{C}^\times)$ *is isomorphic to the semidirect product* $\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$.

Exercise 24.13. Prove Theorem 24.12. ◇

25. GL$(2, \mathbb{C})$

25.1. **From automorphisms to matrices.** Linear fractional transformations have striking resemblances to two-by-two matrices. Here is a partial dictionary:

$$\begin{aligned} f : z &\mapsto \frac{az + b}{cz + d} && \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A_b : z &\mapsto z + b && \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ M_{a^2} : z &\mapsto a^2 z && \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \\ V : z &\mapsto 1/z && \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

Before justifying the dictionary, we define several groups.

25.2. **The linear groups.**

Definition 25.3.

- GL$(2, \mathbb{C})$ is the *general linear group* over $\mathbb{C}$: the group of invertible two-by-two matrices with complex entries.
- SL$(2, \mathbb{C})$ is the *special linear group* over $\mathbb{C}$: the subgroup of GL$(2, \mathbb{C})$ of matrices with determinant equal to one.

- $\mathbb{C}^\times \cdot \mathrm{Id}$ is the subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of non-zero multiples of the identity.
- The *projectivisations* of the linear groups are:

$$\mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C}) \,/\, \mathbb{C}^\times \cdot \mathrm{Id}$$
$$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C}) \,/\, \pm \mathrm{Id}$$

We can also replace $\mathbb{C}$ everywhere by $\mathbb{R}$; this gives the groups $\mathrm{GL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{PGL}(2,\mathbb{R})$, and $\mathrm{PSL}(2,\mathbb{R})$. $\diamondsuit$

Exercise 25.4. From the definitions we have inclusions:

$$\mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C}) \qquad \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{R})$$

Prove that these induce well-defined group homomorphisms:

$$\rho_\mathbb{C}\colon \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C}) \qquad \rho_\mathbb{R}\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R})$$

Prove that $\rho_\mathbb{C}$ is an isomorphism but $\rho_\mathbb{R}$ is not. [In fact, $\rho_\mathbb{R}$ realises $\mathrm{PSL}(2,\mathbb{R})$ as an index two subgroup of $\mathrm{PGL}(2,\mathbb{R})$.] $\diamondsuit$

Exercise 25.5. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{GL}(2,\mathbb{C})$. Define $f_A(z) = (az+b)/(cz+d)$. Define $\rho\colon \mathrm{PGL}(2,\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$ by $\rho([A]) = f_A$. Prove that ρ a well-defined isomorphism of groups. $\diamondsuit$

25.6. **The unitary groups.** We take

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We say that J has *signature* $(1,1)$: one eigenvalue equal to 1 and one eigenvalue equal to -1. Similarly, we say that Id has signature $(2,0)$ (and $-\mathrm{Id}$ has signature $(0,2)$). Suppose that $A \in \mathrm{GL}(2,\mathbb{C})$ is a matrix. We set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and define} \quad A^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

The matrix A^* is the *conjugate transpose* of A.

Definition 25.7.

- $\mathrm{U}(1,1)$ is the *unitary group* with *signature* $(1,1)$ over $\mathbb{C}$: that is, the subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of the matrices A so that $A^*JA = J$.
- $\mathrm{SU}(1,1)$ is the *special unitary group* over $\mathbb{C}$: the subgroup of $\mathrm{U}(1,1)$ of matrices with determinant equal to one.
- $S^1 \cdot \mathrm{Id}$ is the subgroup of $\mathrm{U}(1,1)$ consisting of unit modulus multiples of the identity.

- The *projectivisations* of the unitary groups are:

$$\mathrm{PU}(1,1) = \mathrm{U}(1,1)\,/\,S^1 \cdot \mathrm{Id}$$
$$\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)\,/\,\pm\mathrm{Id}$$

In similar fashion we define $\mathrm{U}(2)$, the unitary group with signature $(2,0)$: the subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of the matrices A so that $A^*A = \mathrm{Id}$. We also define, as above, the subgroup and quotient groups $\mathrm{SU}(2)$, $\mathrm{PU}(2)$, and $\mathrm{PSU}(2)$. ◇

Exercise 25.8. Prove that $\mathrm{U}(1,1)$ and $\mathrm{U}(2)$ are groups. ◇

Exercise 25.9. From the definitions we have inclusions:

$$\mathrm{SU}(1,1) \to \mathrm{U}(1,1) \qquad \mathrm{SU}(2) \to \mathrm{U}(2)$$

Prove that these induce well-defined group homomorphisms:

$$\rho_{(1,1)}\colon\ \mathrm{PSU}(1,1) \to \mathrm{PU}(1,1) \qquad \rho_{(2)}\colon\ \mathrm{PSU}(2) \to \mathrm{PU}(2)$$

Prove that both $\rho_{(1,1)}$ and $\rho_{(2)}$ are isomorphisms. ◇

Exercise 25.10.

- Prove that every $A \in \mathrm{SU}(1,1)$ has the form $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.
- Prove that every $A \in \mathrm{SU}(2)$ has the form $A = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$. ◇

25.11. **Three-transitivity of $\mathbf{A\scriptstyle UT}(\hat{\mathbb{C}})$.** The action of $\mathrm{Aut}(\hat{\mathbb{C}})$ on $\hat{\mathbb{C}}$ is *(uniquely) three-transitive* as follows.

Exercise 25.12. Suppose that u, v, and w are distinct points of $\hat{\mathbb{C}}$. Then there is a unique linear fractional transformation sending u, v, and w (in that order) to ∞, 0, and 1 (in that order). ◇

25.13. **Subgroups of $\mathbf{A\scriptstyle UT}(\hat{\mathbb{C}})$.** Recall that in Exercise 25.5 we gave an isomorphism $\rho\colon\ \mathrm{PGL}(2,\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$. We use this to explore certain subgroups of $\mathrm{Aut}(\hat{\mathbb{C}})$.

Suppose that $U \subset \mathbb{C}$ is a domain (or $U = \hat{\mathbb{C}}$). Suppose that $X \subset U$. Define

$$\mathrm{Aut}_X(U) = \{f \in \mathrm{Aut}(U) \mid f(X) = X\}$$

This is the *(setwise) stabiliser* of X in $\mathrm{Aut}(U)$. When $X = \{p\}$ is a singleton, we simplify the notation to just $\mathrm{Aut}_p(U)$.

Exercise 25.14. Prove that $\mathrm{Aut}_\infty(\hat{\mathbb{C}})$ (the stabiliser of infinity) is isomorphic to $\mathrm{Sim}(\mathbb{C})$. ◇

We define the *extended line* $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ in similar fashion to the extended plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The inclusion of $\mathbb{R}$ into $\mathbb{C}$ induces an inclusion of $\hat{\mathbb{R}}$ into $\hat{\mathbb{C}}$, sending infinity to itself. Combining the above definitions and notations we have that $\mathrm{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}})$ is the subgroup of automorphisms (of $\hat{\mathbb{C}}$) preserving $\hat{\mathbb{R}}$ setwise.

Recall our notations for the unit circle S^1, the unit disc $\mathbb{D}$, and the upper half plane $\mathbb{H}$:

$$S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$$
$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$$
$$\mathbb{H} = \{z \in \mathbb{C}\colon \textsc{Imag}(z) > 0\}$$

Exercise 25.15. Suppose that $\rho\colon \mathrm{PGL}(2, \mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$, defined by $\rho([A]) = f_A$, is the isomorphism given in Exercise 25.5.

- Prove that $\rho(\mathrm{PGL}(2, \mathbb{R})) = \textsc{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}})$.
- Prove that the resulting action of $\mathrm{PGL}(2, \mathbb{R})$ on $\hat{\mathbb{R}}$ is three-transitive.
- Prove that the resulting action of $\mathrm{PSL}(2, \mathbb{R})$ on $\hat{\mathbb{R}}$ is not three-transitive. Instead it is transitive on *anticlockwise triples*. Here an ordered triple (u, v, w) of points in $\hat{\mathbb{R}}$ is *anticlockwise* if the orientation it induces on $\hat{\mathbb{R}}$ agrees with the orientation induced by the upper half plane $\mathbb{H}$.
- Prove that $\rho(\mathrm{PSL}(2, \mathbb{R})) = \textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$. ◇

Exercise 25.16.

- Find some $f \in \textsc{Aut}(\hat{\mathbb{C}})$ so that $f(\mathbb{D}) = \mathbb{H}$.
- Prove that, for the same f, we have $f(S^1) = \hat{\mathbb{R}}$
- Prove that, for the same f, we have

$$\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}}) = f^{-1} \circ \textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}}) \circ f$$

  We say that the subgroups $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$ and $\textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$ are *conjugate* inside of $\textsc{Aut}(\hat{\mathbb{C}})$. In particular, they are isomorphic.
- Prove that, for the same f, we have

$$\textsc{Aut}_{S^1}(\hat{\mathbb{C}}) = f^{-1} \circ \textsc{Aut}_{\hat{\mathbb{R}}}(\hat{\mathbb{C}}) \circ f$$ ◇

Exercise 25.17. Suppose that $\rho\colon \mathrm{PGL}(2, \mathbb{C}) \to \textsc{Aut}(\hat{\mathbb{C}})$, defined by $\rho([A]) = f_A$, is the isomorphism given in Exercise 25.5.

- Prove that $\rho(\mathrm{PU}(1, 1)) = \textsc{Aut}_{S^1}(\hat{\mathbb{C}})$.
- Prove that the resulting action of $\mathrm{PU}(1, 1)$ on S^1 is three-transitive.
- Prove that the resulting action of $\mathrm{PSU}(1, 1)$ on S^1 is not three-transitive. Instead it is transitive on *anticlockwise triples*. Here an ordered triple (u, v, w) of points in S^1 is *anticlockwise* if the orientation it induces on S^1 agrees with the orientation induced by the unit disc $\mathbb{D}$.
- Prove that $\rho(\mathrm{PSU}(1, 1)) = \textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$. ◇

26. The unit disc

Recall our notation for the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$$

26.1. **Schwarz's lemma.** For a history of the lemma (first found in work of Carathéodory, who attributes it to Schmidt) we refer to [6, page 272].

Lemma 26.2. *Suppose that $f \colon \mathbb{D} \to \mathbb{D}$ is holomorphic. Suppose that $f(0) = 0$. Then, for all $z \in \mathbb{D}$, we have $|f(z)| \leq |z|$. Also, $|f'(0)| \leq 1$. Furthermore, if there is some $z \neq 0$ with $|f(z)| = |z|$, or if $|f'(0)| = 1$, then f is a rotation.*

Proof. If f is constant then it is identically zero and we are done.

Suppose that f is non-constant in $\mathbb{D}$. Factoring (Lemma 13.11) we obtain a holomorphic function $g \colon \mathbb{D} \to \mathbb{C}$ so that $f(z) = zg(z)$. Note that $g(0) = f'(0)$.

Fix any $z_0 \neq 0$ in the open disc. Fix any r so that $|z_0| < r < 1$. We now apply the maximum modulus principle (Corollary 22.3) to g and the region $\overline{B(0;r)}$. This gives us some $z_r \in C(0;r)$ as follows:

$$|g(z_0)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} < \frac{1}{r}$$

As r tends to 1 (from below) we deduce that $|g(z_0)| \leq 1$. Since z_0 was arbitrary, we deduce the inequality $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$, as desired.

Note that the derivative $f'(0)$ is defined via the limit of difference quotients:

$$\frac{f(z) - f(0)}{z - 0} = \frac{f(z)}{z} = g(z)$$

Since $|g(z)| \leq 1$, we deduce that $|f'(0)| \leq 1$.

Suppose now that either

- for some $z_0 \in \mathbb{D}^\times$ we have $|f(z_0)| = |z_0|$ or
- $z_0 = 0$ and $|f'(0)| = 1$.

In either case, $|g(z)|$ has a local maximum at z_0. Again applying the maximum modulus principle (Theorem 22.2) we deduce that g is constant and has modulus one. So f is a rotation, as desired. □

26.3. **Automorphism of the unit disc.**

Lemma 26.4. *Suppose that a and b lie in $\mathbb{C}$, with $|a|^2 - |b|^2 = 1$. Then the rational map $f(z) = (az + b)/(\bar{b}z + \bar{a})$ is an automorphism of $\mathbb{D}$. Furthermore, all automorphisms of $\mathbb{D}$ are of this form.*

Said another way: $\mathrm{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$ is isomorphic to $\mathrm{Aut}(\mathbb{D})$.

Proof of Lemma 26.4. Suppose that f has the above form. We take

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

So A lies in $\mathrm{SU}(1,1)$. We take $f_A = \rho(A)$ to be the resulting element of $\mathrm{AUT}(\hat{\mathbb{C}})$. Thus, by Exercise 25.17, the automorphism f_A preserves $\mathbb{D}$. Also, we have that $f = f_A|\mathbb{D}$. So f is an automorphism of $\mathbb{D}$.

Suppose now that g is an automorphism of $\mathbb{D}$. Suppose that $g(0) = p$. Let $h(z) = (z-p)/(-\bar{p}z+1)$. So h is an element of $\mathrm{AUT}_{\mathbb{D}}(\hat{\mathbb{C}})$ and thus is an automorphism of $\mathbb{D}$ as above. So $F = h \circ g$ is also an automorphism and, moreover, fixes zero. Let G be the given automorphism so that $F \circ G = \mathrm{Id}_{\mathbb{D}}$. Thus G also fixes zero. Applying Lemma 26.2 we find that $|F'(0)| \leq 1$ and $|G'(0)| \leq 1$.

The chain rule tells us that $F'(G(0)) \cdot G'(0) = 1$. Since $G(0) = 0$, taking moduli gives $|F'(0)| \cdot |G'(0)| = 1$. We deduce that $|F'(0)| = |G'(0)| = 1$. Applying Lemma 26.2 once again we deduce that F is a rotation. Pick $\lambda \in S^1 \in \mathbb{R}$ so that $F(z) = \lambda^2 \cdot z$.

Since $h \circ g = F$ we have that $g = h^{-1} \circ F$. Since $\bar{\lambda} = \lambda^{-1}$ we have:

$$g(z) = \frac{\lambda^2 z + p}{\bar{p}\lambda^2 z + 1} = \frac{\lambda z + \bar{\lambda} p}{\lambda \bar{p} z + \bar{\lambda}}$$

Note that $1 - |p|^2$ is positive. We define

$$A = \frac{1}{\sqrt{1-|p|^2}} \begin{pmatrix} \lambda & \bar{\lambda} p \\ \lambda \bar{p} & \bar{\lambda} \end{pmatrix}$$

Note that $\det(A) = 1$ (so A lies in $\mathrm{SU}(1,1)$). Since $\rho(A) = g$ we are done. □

26.5. **Many examples.** We end with a selection of conformal equivalences.

Example 26.6. The unit disc $\mathbb{D}$ and the upper half plane $\mathbb{H}$ are conformally equivalent via

$$f(z) = \frac{z+i}{iz+1}$$ ◇

Example 26.7. The upper half plane $\mathbb{H}$ and the cut plane $\mathbb{C} - [0, \infty)$ are conformally equivalent via $f(z) = z^2$. ◇

Example 26.8. The quarter plane $Q = \{z \in \mathbb{C} | \mathrm{REAL}(z) > 0, \mathrm{IMAG}(z) > 0\}$ and the upper half plane $\mathbb{H}$ are conformally equivalent via $f(z) = z^2$. ◇

Example 26.9. The domain $U = \mathbb{C} - \overline{B(0;1)}$ and the slit plane $\mathbb{C} - [-1, 1]$ are conformally equivalent via

$$f(z) = \frac{1}{2}(z + \frac{1}{z})$$ ◇

Example 26.10. The lune $L = \{z \in \mathbb{C} \mid \text{Real}(z) > 0, |z| < 1\}$ and the quarter plane $Q = \{z \in \mathbb{C} \mid \text{Real}(z) > 0, \text{Imag}(z) > 0\}$ are conformally equivalent via

$$f(z) = \frac{z+i}{iz+1}$$ ◇

Example 26.11. The punctured disc $\mathbb{D}^\times$ and the domain $U = \mathbb{C} - \overline{B(0;1)}$ are conformally equivalent via $f(z) = 1/z$. ◇

27. The Riemann mapping theorem

27.1. **Montel's theorem.** We give a simplified version of Montel's theorem.

Lemma 27.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $(f_n : U \to \mathbb{D})_n$ is a sequence of holomorphic functions. Then there is a subsequence $(f_{n_k})_k$ that converges (uniformly on compact subsets of U) to a holomorphic function $f : U \to \mathbb{C}$.*

Proof. Fix $z_0 \in U$. Fix $r_0 > 0$ so that $B(z_0; r_0) \subset U$. Each f_n has a series expansion $A_n = (a_{m,n})$ in $B(z_0; r_0)$. Note that $|f_n(z)| < 1$. So we apply Lemma 12.5 and deduce that

$$|a_{m,n}| \leq 1/r_0^m$$

for all m and n. Passing to subsequences countably many times, and then taking the diagonal, we can arrange for the coefficients to converge. The limit is a function $g_0 : B(z_0; r_0) \to \mathbb{C}$. Since g_0 is defined by a power series, it is holomorphic.

Since U is open, it is the union of countably many open discs of the form $B(z_k; r_k)$. Applying the argument above countably many times (passing to subsequences each time), and again taking a diagonal, gives the desired converging subsequence and limiting holomorphic function. □

27.3. **Hurwitz's lemma.**

Lemma 27.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, f_n : U \to \mathbb{C}$ are holomorphic. Suppose that f_n converges (uniformly on compact subsets of U) to f. Suppose that f is non-constant. Suppose that z_0 is any point of U, and $w_0 = f(z_0)$. Then there is some N so that, for all $n \geq N$, there are $z_n \in U$ so that*

- $z_n \to z_0$
- $f_n(z_n) = w_0$.

Proof. Define $g(z) = f(z) - w_0$. Define $g_n(z) = f_n(z) - w_0$. Since f is non-constant, we have that g is non-constant. Thus $Z(g)$ is isolated (Lemma 14.7). Also, the functions g_n converge (uniformly on compact sets) to g.

Fix $r > 0$ so that

- $\overline{B(z_0;r)} \subset U$ and
- $\overline{B(z_0;r)} \cap Z(g) = \{z_0\}$.

Define

$$2\epsilon = \min\{|g(z)| \colon z \in C(z_0;r)\}$$

Note that $\epsilon > 0$ as $C(z_0;r)$ is compact, g is continuous, and g has no zeros on $C(z_0;r)$.

Pick N so that, for all $n \geq N$ and for all $z \in \overline{B(z_0;r)}$, we have

$$|g(z) - g_n(z)| < \epsilon$$

Fix $n \geq N$. So

$$\epsilon > |g(z_0) - g_n(z_0)| = |g_n(z_0)|$$

Also, for $z \in C(z_0;r)$, we have

$$\begin{aligned} |g_n(z)| &= |g(z) + g_n(z) - g(z)| \\ &\geq |g(z)| - |g_n(z) - g(z)| \\ &> 2\epsilon - \epsilon = \epsilon \end{aligned}$$

Thus $|g_n|$ has a local minimum in $B(z_0;r)$. Applying the minimum modulus principle (Theorem 22.5) we find that $g_n(z_n) = 0$ for some z_n in $B(z_0;r)$. Thus $f_n(z_n) = w_0$.

Repeating the argument with smaller r ensures that the points z_n tend to z_0, as desired. □

Corollary 27.5. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f, f_n \colon U \to \mathbb{C}$ are holomorphic. Suppose that $f_n \to f$ uniformly on compact subsets of U. Suppose that all of the f_n are injective. Then f is injective.*

Proof. We prove the contrapositive. Suppose that z_0 and z_0' are distinct points of U with $f(z_0) = f(z_0')$. Applying Lemma 27.4 twice, we have some N so that, for all $n \geq N$, there are points z_n and z_n' so that

- $|z_0 - z_n|$ and $|z_0' - z_n'|$ are less than $|z_0 - z_0'|/2$ and
- $f_n(z_n) = f(z_0)$ and $f_n(z_n') = f(z_0')$.

So z_n and z_n' are distinct and $f_n(z_n) = f_n(z_n')$. So f_n is not injective. □

27.6. **The Riemann mapping theorem.** Our statement of the Riemann mapping theorem is modelled on a more extensive version given by Remmert [7, page 180].

Theorem 27.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Then the following are equivalent:*

(1) $U = \mathbb{C}$ or U is conformally equivalent to $\mathbb{D}$.
(2) U is homeomorphic to $\mathbb{D}$.
(3) U has trivial first homology ($H_1(U) \cong 0$).

Proof. We prove that the last implies the first. If $U = \mathbb{C}$ we are done. So suppose that U is a domain missing some point of $\mathbb{C}$ and having trivial first homology.

Claim. The domain U is conformally equivalent to a domain $V \subset \mathbb{D}$. Furthermore, we may assume that zero lies in V.

Proof. Starting with U we produce a sequence of domains with each conformally equivalent to the previous. Since U has trivial first homology, Exercise 19.10 implies that all of the domains in our sequence have trivial first homology.

Fix some $z_0 \in \mathbb{C} - U$. Define $f(z) = z - z_0$. Take $U' = f(U)$. So U' is a domain, does not contain zero, and is equivalent to U via f. We replace U by U' and reuse the notation U.

Applying Theorem 20.4, we take $g\colon U \to \mathbb{C}$ to be a branch of the logarithm. Define $f\colon U \to \mathbb{C}$ by $f(z) = \textsc{Exp}(\frac{1}{2} \cdot g(z))$. Thus f is a branch of the square root. So, for all $z \in U_1$, we have $(f(z))^2 = z$. Suppose that $f(z) = f(w)$. Thus $z = (f(z))^2 = (f(w))^2 = w$. So f is injective. Take $U' = f(U)$. So U' is a domain, does not contain zero, and is equivalent to U via f. Also, a point w lies in U' if and only if $-w$ does not lie in U'. We replace U by U' and reuse the notation U.

Fix any $z_0 \in U$. Pick any $\epsilon > 0$ so that $\overline{B(z_0; \epsilon)} \subset U$. We deduce that $\overline{B(-z_0; \epsilon)}$ is disjoint from U. Define $f\colon U \to \mathbb{C}$ by $f(z) = \epsilon/(z + z_0)$. Note that $|f(z)| < 1$ for all $z \in U$. Take $U' = f(U)$. So $U' \subset \mathbb{D}$ and is conformally equivalent to U via f. We replace U by U' and reuse the notation U.

Fix any z_0 in U. Define $f\colon \mathbb{D} \to \mathbb{D}$ by $f(z) = (z - z_0)/(-\bar{z}_0 z + 1)$. So f is an automorphism of $\mathbb{D}$. Take $V = f(U)$. So we have $0 \in V \subset \mathbb{D}$; also V is conformally equivalent to U. □

We replace our original domain by V (and again call it U, so simplify our notation).

Claim. Suppose that $U \subset \mathbb{D}$ is a domain. Suppose that zero lies in U. Suppose that U is not all of $\mathbb{D}$. Then there is a domain $V \subset \mathbb{D}$ and a biholomorphic mapping $H\colon U \to V$ so that:

- $H(0) = 0$ and
- $|H'(0)| > 1$.

Proof. Suppose that $-p$ is a point of $\mathbb{D} - U$. Define $f\colon \mathbb{D} \to \mathbb{D}$ by $f(z) = (z + p)/(\bar{p}z + 1)$. Let $U' = f(U)$. Note that $U' \subset \mathbb{D}$ contains p, does not contain zero, and is conformally equivalent to U via f.

Applying Theorem 20.4, we take $g\colon U' \to \mathbb{D}$ to be a branch of the square root. In a small abuse of notation we simply write $g(z) = \sqrt{z}$. As above, g is a biholomorphism to its image $U'' = g(U')$. Note that $U'' \subset \mathbb{D}$ contains $g(p) = \sqrt{p}$, does not contain zero, and is conformally equivalent to U' via g.

Define $h\colon U'' \to \mathbb{D}$ by $h(z) = (z - \sqrt{p})/(-\sqrt{\bar{p}}z + 1)$. Define $U''' = h(U'')$. Note that $U''' \subset \mathbb{D}$ contains zero and is conformally equivalent to U'' via h.

Take $H = h \circ g \circ f$; since H is a composition of biholomorphic mappings it is itself biholomorphic. We have

$$H(0) = h(g(f(0))) = h(g(p)) = h(\sqrt{p}) = 0$$

We next estimate $|H'(0)|$. To recap, we have:

$$h(z) = \frac{z - \sqrt{p}}{-\sqrt{\bar{p}}z + 1} \qquad g(z) = \sqrt{z} \qquad f(z) = \frac{z + p}{\bar{p}z + 1}$$

Taking derivatives we have:

$$h'(z) = \frac{1 - |p|}{(-\sqrt{\bar{p}}z + 1)^2} \qquad g'(z) = \frac{1}{2\sqrt{z}} \qquad f'(z) = \frac{1 - |p|^2}{(\bar{p}z + 1)^2}$$

So we have

$$h'((g \circ f)(0)) = \frac{1}{1 - |p|} \qquad g'(f(0)) = \frac{1}{2\sqrt{p}} \qquad f'(0) = 1 - |p|^2$$

Thus

$$H'(0) = (h \circ g \circ f)'(0) = \frac{1}{1 - |p|} \cdot \frac{1}{2\sqrt{p}} \cdot (1 - |p|^2) = \frac{1 + |p|}{2\sqrt{p}}$$

This has modulus:

$$\frac{|p|^{1/2} + |p|^{-1/2}}{2} > 1$$

□

Suppose now that $U \subset \mathbb{D}$ is a domain, with $0 \in U$. We define

$$\mathcal{H} = \{f \colon U \to \mathbb{D} \mid f(0) = 0,\ f \text{ holomorphic},\ f \text{ injective}\}$$

Suppose that f is in $\mathcal{H}$. The open mapping theorem (in the form of Proposition 23.3) implies that f is a conformal equivalence between U and $f(U)$.

We define the extremal quantity D as follows:

$$D = \sup\{|f'(0)| \colon f \in \mathcal{H}\}$$

Suppose that $r > 0$ has $\overline{B(0;r)} \subset U$. By Lemma 12.5 we have $f'(0) \leq 1/r$. Thus $D \leq 1/r$. So D is bounded above. Also, Id_U lies in $\mathcal{H}$. Thus $D \geq 1$. So D is bounded below.

Let $(f_n)_n$ be any sequence in $\mathcal{H}$ so that $|f_n'(0)|$ tends to D from below. Applying Lemma 27.2 we pass to a subsequence that converges (uniformly on compact subsets) to some holomorphic function $F \colon U \to \mathbb{C}$. Note that $F(0) = 0$. Also, the image $F(U)$ lies in $\overline{\mathbb{D}}$. Applying Corollary 27.5 we find that F is injective. In particular, F is non-constant. So, by the open mapping theorem (Theorem 22.7) we have that $F(U)$ is open. So $F(U)$ lies in $\mathbb{D}$. Thus F is a element of $\mathcal{H}$.

Claim. $F(U) = \mathbb{D}$

Proof. For a contradiction, suppose that $F(U)$ is not equal to $\mathbb{D}$. Then the previous claim gives a domain $V \subset \mathbb{D}$ and a biholomorphic mapping $H \colon F(U) \to V$ so that $H(0) = 0$ and $|H'(0)| > 1$. Thus $H \circ F$ lies in $\mathcal{H}$. We compute:

$$|(H \circ F)'(0)| = |H'(0)| \cdot |F'(0)| > D$$

However, this contradicts the maximality of D. □

This completes the proof of the Riemann mapping theorem (27.7). □

Solution 1.4. Recall that $\cos\colon \mathbb{R} \to \mathbb{R}$ is continuous; also $\cos(0) = 1$ and $\cos(\pi) = -1$. Since $C^2 + S^2 = 1$ we have that $C \in [-1, 1]$. The intermediate value theorem gives some $\theta \in [0, \pi]$ so that $\cos(\theta) = C$. Note that $\cos(-\theta) = C$ as well. We deduce that $\sin^2(\theta) = S^2$. Taking square roots (and replacing θ by $-\theta$ if needed) proves that θ lies in $\mathrm{Arg}(z)$.

Since cosine and sine are 2π–periodic, we have that $\theta + 2\pi k$ lies in $\mathrm{Arg}(z)$ for all $k \in \mathbb{Z}$. Suppose that η is an element of $\mathrm{Arg}(z)$. Using the addition we have the following.

$$\begin{aligned}\cos(\theta - \eta) &= \cos(\theta)\cos(-\eta) - \sin(\theta)\sin(-\eta)\\ &= \cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)\\ &= C^2 + S^2\\ &= 1\end{aligned}$$

Recall that 2π is the least period of cosine. Thus $\theta - \eta$ lies in $2\pi\mathbb{Z}$, as desired. □

Solution 1.5. Suppose that $z = x + iy$. Suppose that $w = a + ib$. So $zw = xa - yb + i(xb + ay)$. We now compute as follows:

$$\begin{aligned}|zw|^2 &= (xa - yb)^2 + (xb + ya)^2\\ &= x^2a^2 + y^2b^2 - 2xyab + x^2b^2 + y^2a^2 + 2xyab\\ &= x^2a^2 + x^2b^2 + y^2a^2 + y^2b^2\\ &= (x^2 + y^2)(a^2 + b^2)\\ &= |z|^2|w|^2\end{aligned}$$

Taking the positive square root of both sides gives $|zw| = |z||w|$, as desired.

By the first part of Exercise 1.4 we may choose θ and η to be arguments of z and w respectively. By the second part of Exercise 1.4 it suffices to prove that $\theta + \eta$ is an argument of zw. We have the following polar forms:

$$z = |z|(\cos(\theta) + i\sin(\theta)) \qquad w = |w|(\cos(\eta) + i\sin(\eta))$$

Taking their product, and applying the addition rule, gives:

$$zw = |z||w|(\cos(\theta + \eta) + i\sin(\theta + \eta))$$

By the above we have $|z||w| = |zw|$. Thus $\theta + \eta$ is an argument of zw, as desired. □

Solution 1.6. Suppose that $z = x + iy$ and $w = r + is$. We compute as follows.

$$\begin{aligned}
|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta) &= \\
&= |z|^2 + |w|^2 - 2|z||w|(\cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)) \\
&= |z|^2 + |w|^2 - 2|z||w|\left(\frac{x}{|z|}\frac{r}{|w|} + \frac{y}{|z|}\frac{s}{|w|}\right) \\
&= |z|^2 + |w|^2 - 2(xr + ys) \\
&= x^2 + y^2 + r^2 + s^2 - 2(xr + ys) \\
&= x^2 - 2xr + r^2 + y^2 - 2ys + s^2 \\
&= (x - r)^2 + (y - s)^2 \\
&= |z - w|^2
\end{aligned}$$

□

Solution 1.8. Suppose that a, b, c, and d are complex numbers. Suppose that f and g are similarities with $f(z) = az+b$ and $g(z) = cz+d$. Then $f \circ g(z) = a(cz + d) + b = acz + (ad + b)$. Since a and c are non-zero, so is ac. Thus $f \circ g$ is again a similarity.

We obtain an identity by taking $a = 1$ and $b = 0$. We make g an inverse for f by setting $c = 1/a$ and $d = -b/a$. The group operation is associative because function composition is associative.

Suppose that $|a| = |c| = 1$. Thus $|ac| = 1$ and $f \circ g$ is a congruence. Also $|1/a| = 1$, so the inverse of f is a congruence.

More generally, if the condition on a is closed under multiplication and inversion, and b is general or zero, then we obtain a subgroup. □

Solution 1.9. If $w = 0$ then $|z - w| < |z|$ becomes $|z| < |z|$, a contradiction. Thus $\textrm{Arg}(w)$ is well-defined. Fix any $\theta \in \textrm{Arg}(z)$. By the pigeonhole principle, there is at least one $\eta \in \textrm{Arg}(w)$ so that $|\theta - \eta| \leq \pi$.

Suppose, for a contradiction, that there are at least two such. Since 2π is the least period of $\textrm{Arg}(w)$, we deduce that $\theta + \pi$ and $\theta - \pi$ both lie in $\textrm{Arg}(w)$. From their polar forms we deduce that z and w lie on a line through the origin, with the origin in-between. Thus $|z - w| = |z| + |w|$. Thus $|z| + |w| < |z|$, a contradiction. Thus there is exactly one $\eta \in \textrm{Arg}(w)$ so that $|\theta - \eta| < \pi$. □

Solution 2.4. Suppose that p is a point of U. Let U_p be the path-component of U that contains p. We now prove the contrapositive: if U_p is not all of U then U is not connected.

Since U is open, there is some $\epsilon > 0$ so that $B(p; \epsilon)$ is contained in U. Since $B(p; \epsilon)$ is path-connected, it is a subset of U_p. Similarly, U_p contains an open neighbourhood of all of its points. So U_p is open.

Define $U' = U - U_p$. Fix any q in U'. Since U is open there is some $\delta > 0$ so that $B = B(q; \delta)$ is contained in U. Note that, since q is not in U_p, we have that B is disjoint from U_p. Thus B is a subset of U'. So U' is open. Thus U is not connected. □

Solution 2.5. If $U = \mathbb{C}$ then any constant $\epsilon > 0$ works. Suppose instead that U is a proper subset of $\mathbb{C}$. Suppose that q lies in $\mathbb{C} - U$. Since distance is continuous and P is compact, there is some point $p \in P$ which is as close as possible to q. Thus writing $d(q, P) = |q - p|$ is well-defined. Suppose, for a contradiction, that (q_n) is a sequence in $\mathbb{C} - U$ so that $d(q_n, P)$ tends to zero. Let p_n be a point of P closest to q_n. Since P is compact, the sequence (q_n) is bounded, so has a convergent subsequence, which we pass to. Since $\mathbb{C} - U$ is closed, the limit q_∞ lies in $\mathbb{C} - U$. We pass to a further subsequence to ensure that (p_n) also converges say to p_∞ in P. But then $|q_\infty - p_\infty| = 0$. Thus $q_n = p_n$. We deduce that P and $\mathbb{C} - U$ intersect, a contradiction. □

Solution 2.16. By Lemma 2.13 and Example 2.11 it suffices to prove that $\int_\gamma z^k \, dz = 0$ for $k \neq -1$. We compute as follows.

$$\begin{aligned}\int_\gamma z^k \, dz &= \int_0^{2\pi} e^{ki\theta} \cdot ie^{i\theta} \, d\theta \\ &= i \int_0^{2\pi} e^{(k+1)i\theta} \, d\theta \\ &= \frac{i}{(k+1)i} \left[e^{(k+1)i\theta} \right]_0^{2\pi} \\ &= \frac{1}{k+1}(1-1) \\ &= 0\end{aligned}$$

Note we here used the fundamental theorem of calculus for the real-valued cosine and sine (not for the complex exponential). □

Solution 2.21. Suppose that δ preserves *orientation*: that is, $\delta(c) = a$ and $\delta(d) = b$. Then the sign of δ' is positive. Suppose that δ reverses orientation: that is, $\delta(c) = b$ and $\delta(d) = a$. Then the sign of δ' is negative.

Suppose first that δ preserves orientation. Then we compute as follows.

$$\begin{aligned}
L(\gamma \circ \delta) &= \int_{\gamma\circ\delta} \sqrt{dx^2 + dy^2} \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2(\delta'(u))^2 + (\gamma_y'(\delta(u)))^2(\delta'(u))^2}\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot |\delta'(u)|\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot \delta'(u)\, du \\
&= \int_a^b \sqrt{(\gamma_x'(t))^2 + (\gamma_y'(t))^2}\, dt \\
&= \int_a^b \sqrt{(\gamma_x'(t))^2 + (\gamma_y'(t))^2}\, dt \\
&= \int_\gamma \sqrt{dx^2 + dy^2} \\
&= L(\gamma)
\end{aligned}$$

If δ reverses orientation, then the two minus signs cancel, and we obtain the same arc-length. □

Solution 2.23. We compute as follows.

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f\, dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leq M \cdot L(\gamma) \qquad \square$$

Solution 3.5. Suppose that $\gamma(s) = x(s) + iy(s)$. We suppress the variable s to obtain $\gamma = x + iy$ and $\gamma' = x' + iy'$. We may also write $r = \sqrt{x^2 + y^2}$. Thus $r'/r = (xx' + yy')/(x^2 + y^2)$. The fundamental theorem of calculus gives $\theta' = (xy' - yx')/(x^2 + y^2)$. We compute:

$$\frac{\gamma'}{\gamma} = \frac{x' + iy'}{x + iy} = \frac{x' + iy'}{x + iy}\frac{(x - iy)}{(x - iy)} <= \frac{xx' + yy' + i(xy' - yx')}{x^2 + y^2} = \frac{r'}{r} + i\theta'$$

Recall that $h = re^{i\theta}/\gamma$. We differentiate:

$$h' = \frac{r'e^{i\theta} + re^{i\theta}i\theta'}{\gamma} - \frac{re^{i\theta}\gamma'}{\gamma^2} = \frac{re^{i\theta}}{\gamma}\left[\frac{r'}{r} + i\theta' - \frac{\gamma'}{\gamma}\right] = 0 \qquad \square$$

Solution 4.7. Fix z_0 in U. Fix $\epsilon > 0$. From Definition 4.2 we have a δ so that if $0 < |z - z_0| < \delta$ then

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

Clearing the denominator we have

$$|(f(z) - f(z_0)) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|$$

Thus

$$\begin{aligned} |f(z) - f(z_0)| &< |f'(z_0)||z - z_0| + \epsilon|z - z_0| \\ &< (|f'(z_0)| + \epsilon)\delta \end{aligned}$$

and f is continuous. □

Solution 4.9. Since f is continuous (by Exercise 4.7) we have that ρ is continuous away from z_0. The limit as z tends to z_0 is zero, by construction and since f is holomorphic. Thus ρ is also continuous at z_0. □

Solution 4.15. We only sketch the proofs – you should provide the missing details.

Note that we can re-choose coordinates so that $z_0 = 0$. Using Theorem 4.13 we have the following.

Suppose that $f = g$. So all derivatives of $f - g$ vanish. Set $c_k = a_k - b_k$. Since $c_k \cdot k! = (f - g)^(k)(0)$ is the k^{th} derivative we are done.

Suppose that $A = B$. Then f and g agree pointwise, so are equal.

Suppose that f_n and g_n are the partial sums. Take any $0 < R' < R$. Then $p \cdot f_n + q \cdot g_n$ converges uniformly on the disc of radius R' to $p \cdot f + q \cdot g$. Thus series coefficents scale and add correctly.

Let $C = (c_n)_n$ where $c_n = \sum_{k+\ell=n} a_k b_\ell$. Let $h_n(z) = \sum_{m \leq n} c_m z^m$. Restricting to any radius R' with $0 < R' < R$ and suppressing the notation for z we have:

$$\begin{aligned} |f_{2n} \cdot g_{2n} - h_{2n}| &\leq |f_{2n} \cdot g_{2n} - f_n g_n| \\ |f_{2n+1} \cdot g_{2n+1} - h_{2n+1}| &\leq |f_{2n+1} \cdot g_{2n+1} - f_n g_n| \end{aligned}$$

The right-hand sides converge to zero as n tends to infinity, proving the result.

Relating the power series for $z^k g(z)$ to that of $g(z)$ is a (very) special case of multiplying power series, dealt with above.

We only sketch the argument.[2] It is convenient to restrict to the case where $a_0 = 1$. (You should check that this is allowed.) We find the coefficients for $g = 1/f$ by first guessing a recurrence relation they satisfy (using the previously given formula for a product). Next we note that f has no zeros on an open disc of positive radius $S > 0$, say. We next prove that $|b_n| \leq 1/S^n$. Thus g has radius of convergence at least S. We finally apply the uniqueness statement. □

Solution 4.16. Applying the geometric series we find

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + z^8 - z^1 0 + \ldots$$

Thus $R = 1$. □

[2] For more detailed accounts see [2, Theorem 9.26, page 239], or [8, Theorem 3.55, page 71], or [4, Proposition 10.16, page 191].

Solution 4.17. We may factor the quadratic into linear terms, realise f as the sum of their reciprocals, apply the geometric series, and add coefficients (appealing to Exercise 4.15. Thus f has a power series about zero. Suppose that $(a_n)_n$ are the coefficients. Since $f(z)\cdot(1+z+z^2) = 1$ we can multiply out, equate coefficients, and find:

$$a_0 = 1, \quad a_1 + a_0 = 0, \quad a_2 + a_1 + a_0 = 0$$

and in general $a_{n+2} + a_{n+1} + a_n = 0$ for $n \geq 0$. By induction, we have

$$a_n = \begin{cases} 1, & \text{if } n = 0 \pmod 3 \\ -1, & \text{if } n = 1 \pmod 3 \\ 0, & \text{if } n = 2 \pmod 3 \end{cases}$$

Thus the radius of convergence is one. □

Solution 4.20. Suppose that $\mathrm{SIN}(z) = 0$. Thus $\mathrm{COS}(z) = \pm 1$.

Suppose that $\mathrm{COS}(z) = 1$. Thus $\mathrm{EXP}(iz) = 1$. Suppose that $z = x + iy$. We now compute using the properties of complex exponential.

$$\begin{aligned} 1 &= \mathrm{EXP}(iz) \\ &= \mathrm{EXP}(ix - y) \\ &= \mathrm{EXP}(ix)\,\mathrm{EXP}(-y) \\ &= (\mathrm{COS}(x) + i\,\mathrm{SIN}(x))\,\mathrm{EXP}(-y) \\ &= (\cos(x) + i\sin(x))\exp(-y) \\ &= \cos(x)\exp(-y) + i\sin(x)\exp(-y) \end{aligned}$$

Equating real and imaginary parts we find that

$$1 = \cos(x)\exp(-y) \quad \text{and} \quad 0 = \sin(x)\exp(-y)$$

Since the exponential has no roots, we find that $\sin(x) = 0$. Thus $x = \pi k$ for some $k \in \mathbb{Z}$. Thus $\cos(x) = \pm 1$. However, $\exp(-y)$ is positive. We deduce that $\cos(x) = 1$ and $y = 0$. Thus $z = x + iy = \pi k$, with k even.

When $\mathrm{COS}(z) = -1$ we find instead that k is odd. □

Solution 4.21. Note that $\mathrm{EXP}(x + iy) = \mathrm{EXP}(x)\,\mathrm{EXP}(iy) = e^x e^{iy}$. Recall that e^x is a positive real number, while e^{iy} is a "direction": a point on the unit circle. If we vary x while fixing y then we obtain a radial arc with direction e^{iy}. If we fix x while varying y then we obtain an arc of the circle, centred on the origin, with radius e^x.

We now take the image of the rectangle R. The horizontal sides both go to the line segment $[a, b]$ in the real axis. The left vertical side goes to the circle $C(0; a)$; the right vertical side goes to the circle $C(0; b)$. These circles cobound the closed annulus A with inner and outer radii equal to a and b, respectively. □

Solution 5.5. Similarities preserve convexity. So it suffices to prove this for the unit disc $\mathbb{D} = B(0; 1)$ and its closure $\overline{\mathbb{D}}$. Suppose that p

and q are points in $\mathbb{D}$. Suppose that $t \in [0, 1]$. We must show that $(1-t)p + tq$ lies in $\mathbb{D}$.

Since p and q lie in $\mathbb{D}$ there is some $r < 1$ so that both $|p|$ and $|q|$ are at most r. We now apply the triangle inequality to find:

$$|(1-t)p + tq| \leq (1-t)|p| + t|q| \leq (1-t)r + tr = r < 1$$

For $\overline{\mathbb{D}}$ we replace the condition $r < 1$ by the weaker condition $r \leq 1$. ☐

Solution 5.6. Suppose that H is the convex hull of S. Suppose that p and q are points of H. Suppose that K is a convex set containing S. Since H So $H \subset K$. So p and q are points of K. Thus $[p, q]$ is a subset of K. Thus $[p, q]$ is a subset of H, as desired. ☐

Solution 6.6. We break $I(\partial T)$ into pieces using its definition and Lemma 2.14:

$$I(\partial T) = I([p, q]) + I([q, r]) + I([r, p])$$

Each term on the right can be broken in two using the midpoints, Lemma 2.14, and the first part of Lemma 2.15:

$$\begin{aligned} I(\partial T) &= I([p, r']) + I([r', q]) \\ &+ I([q, p']) + I([p', r]) \\ &+ I([r, q']) + I([q', p]) \end{aligned}$$

We rearrange terms:

$$\begin{aligned} I(\partial T) &= I([p, r']) + I([q', p]) \\ &+ I([r', q]) + I([q, p']) \\ &+ I([p', r]) + I([r, q']) \end{aligned}$$

We add and subtract terms:

$$\begin{aligned} I(\partial T) &= I([p, r']) + I([r', q']) + I([q', p]) \\ &+ I([r', q]) + I([q, p']) + I([p', r']) \\ &+ I([q', p']) + I([p', r]) + I([r, q']) \\ &- I([r', q']) + I([p', r']) + I([q', p']) \end{aligned}$$

We apply the second part of Lemma 2.15 and rearrange:

$$\begin{aligned} I(\partial T) &= I([p, r']) + I([r', q']) + I([q', p]) \\ &+ I([q, p']) + I([p', r']) + I([r', q]) \\ &+ I([r, q']) + I([q', p']) + I([p', r]) \\ &+ I([p', q']) + I([q', r']) + I([r', p']) \end{aligned}$$

We now use Lemma 2.14 and the definitions:

$$I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$$

This proves the claim. ☐

Solution 6.7. We cut R along a diagonal into a pair of triangles S and T. By Lemma 6.3 we have that the integrals of f over ∂S and over ∂T both vanish. Adding these integrals, and applying Lemma 2.14, we find that the integral of f over ∂R also vanishes. □

Solution 7.2. By assumption the closed contour ∂P is a finite union of line segments (γ_k). Here $\gamma_k \cap \gamma_{k+1}$ is the endpoint of the former and the startpoint of the latter. Let L_k be the line in $\mathbb{C}$ containing γ_k. Each component of $P - \cup L_k$ is a finite-sided convex polygon. Fix one such, say P_ℓ. Fix also some vertex v of ∂P_ℓ; we connect v to all of the other vertices of P_ℓ by line segments. This realises P as a union of triangles with disjoint interiors.

Let (T_k) be this finite collection of triangles. For each k we orient T_k positively: that is, following Definition 5.7 we order the vertices of T_k so that ∂T_k is oriented anticlockwise. By Lemma 6.3 the integral of f about ∂T_k is zero. Thus

$$\sum_k \int_{\partial T_k} f\,dz = 0$$

On the other hand, where two distinct triangles T_k and T_ℓ intersect, they intersect in a single line segment; the boundaries of T_k and T_ℓ give this segment opposite orientations. Thus, applying Lemma 2.14 and Lemma 2.15, we have

$$\sum_k \int_{\partial T_k} f\,dz = \int_{\partial P} f\,dz$$

So the latter vanishes and we are done. □

Solution 7.7. By Example 2.11, the integral of f about the unit circle does not vanish. Thus, by Lemma 7.6, the function f does not have a primitive in $\mathbb{C}^\times$. □

Solution 8.4. They are $\sigma^3|[v_1, v_2, v_3]$, $\sigma^3|[v_0, v_2, v_3]$, $\sigma^3|[v_0, v_1, v_3]$, and $\sigma^3|[v_0, v_1, v_2]$. □

Solution 8.8. Since the boundary operators are homomorphisms, it suffices to fix a single singular two-simplex $\sigma^2 \colon \Delta^2 \to X$ and then to prove that $\partial_1\partial_2\sigma^2 = 0$. We compute as follows:

$$\begin{aligned}
(\partial_1 \circ \partial_2)\sigma^2 &= \partial_1(\partial_2\sigma^2) \\
&= \partial_1(\sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]) \\
&= \partial_1(\sigma^2|[v_1, v_2]) - \partial_1(\sigma^2|[v_0, v_2]) + \partial_1(\sigma^2|[v_0, v_1]) \\
&= (\sigma^2|[v_2] - \sigma^2|[v_1]) \\
&\quad - (\sigma^2|[v_2] - \sigma^2|[v_0]) \\
&\quad + (\sigma^2|[v_1] - \sigma^2|[v_0]) \\
&= 0
\end{aligned}$$

□

Solution 8.15. Along $[v_0, v_2]$ we have $x_1 = 0$ and $x_2 \in [0, 1]$ (with that orientation). So $c|[v_0, v_2]$ is the function $x_2 \mapsto 1 - x_2$.

Along $[v_1, v_2]$ we have $x_0 = 0$, we have $x_1 + x_2 = 1$, and we have $x_2 \in [0, 1]$ (with that orientation). We deduce that $x_1 = 1 - x_2$. So $c|[v_1, v_2]$ is, again, the function $x_2 \mapsto 1 - x_2$. □

Solution 9.3. Note that γ is a closed contour in $\mathbb{C}^\times$. Note that $f(z) = 1/z$ is holomorphic in $\mathbb{C}^\times$. Recall from Example 2.11 that $\int_\gamma dz/z \neq 0$. Now apply the contrapositive of Theorem 9.2: since the contour integral does not vanish γ is not a one-boundary. □

Solution 12.6. This follows from the definitions and the ML–inequality (2.22). Here is the computation:

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \\
&\leq \frac{1}{2\pi} \int_C \left| \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \\
&\leq \frac{1}{2\pi} \int_C \frac{M}{r^{n+1}}\, |dz| \\
&\leq \frac{M}{2\pi r^{n+1}} 2\pi r \\
&\leq \frac{M}{r^n}
\end{aligned}$$

□

Solution 12.8. Set $F_N(w) = \sum_{n=0}^{N} a_n (w - z_0)^n$. We repeat the computation, replacing F by F_N.

$$\begin{aligned}
F_N(w) &= \sum_{n=0}^{N} a_n (w - z_0)^n && \text{definition } F_N(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right] (w - z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z - z_0)^{n+1}} (w - z_0)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\sum \left(\frac{w - z_0}{z - z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\frac{1 - \left(\frac{w - z_0}{z - z_0} \right)^{N+1}}{1 - \left(\frac{w - z_0}{z - z_0} \right)} \right] dz && \text{geometric sum} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - w}\, dz - \frac{1}{2\pi i} \int \frac{f(z)}{z - w} \left(\frac{w - z_0}{z - z_0} \right)^{N+1} dz && \text{simplify} \\
&= f(w) - \frac{1}{2\pi i} \int \frac{f(z)}{z - w} \left(\frac{w - z_0}{z - z_0} \right)^{N+1} dz && \text{Theorem 11.1}
\end{aligned}$$

Note that w lies inside of $B(z_0; r)$. Recall that C is the boundary of $\overline{B(z_0; r)}$. Thus

$$M' = \max\{|f(z)/(z-w)| : z \in C\}$$

is well-defined. Again with z in C we have that $\rho = |w - z_0|/|z - z_0|$ is constant and less than one. Rearranging and applying the ML–inequality (2.22) we have:

$$|f(w) - F_N(w)| \leq \frac{1}{2\pi} M' \rho^{N+1} 2\pi r \leq M' r \rho^{N+1}$$

Note that M' and r are independent of N. Finally, ρ^{N+1} tends to zero as N tends to infinity. □

Solution 13.5. Recall that U is path-connected (Exercise 2.4). So, fix a path $\gamma\colon [0,1] \to U$ connecting z_0 to z_1. By Exercise 2.5 there is some $\epsilon > 0$ so that the neighbourhood $N(\gamma; \epsilon)$ lies in U. Note that γ is continuous on an compact set, so it is uniformly continuous. So there is a $\delta > 0$ with the following property: if $s, t \in [0,1]$ have $|s - t| < \delta$ then $|\gamma(s) - \gamma(t)| < \epsilon/2$.

We now define a sequence of points $(w_k)_{k=0}^N$. Take $t_0 = 0$ and take $w_0 = \gamma(t_0) = z_0$. Suppose that we have obtained $t_\ell \in [0,1]$ and $w_\ell = \gamma(t_\ell)$, with $t_\ell < 1$. So we take $t_{\ell+1} \in [0,1]$ be the maximum of $\gamma^{-1}(\overline{B(w_\ell; \epsilon/2)})$ and take $w_{\ell+1} = \gamma(t_{\ell+1})$. Since $\gamma(t_\ell) = w_\ell$, we have that t_ℓ lies in $\gamma^{-1}(\overline{B(w_\ell; \epsilon/2)})$. Thus $t_\ell < t_{\ell+1}$. If $t_{\ell+1} = 1$ then $w_{\ell+1} = z_1$. In this case we take $N = \ell + 1$ and the construction is complete.

Suppose instead that $t_{\ell+1} < 1$. In this case $|w_{\ell+1} - w_\ell| = \epsilon/2$. Thus $t_{\ell+1} - t_\ell \geq \delta$. This holds for all k. We deduce that $\ell < 1/\delta$. In particular, the construction ends after at most $1 + 1/\delta$ steps. Thus we have produced $(w_k)_{k=0}^N$ in U so that

- $w_0 = z_0$,
- $w_N = z_1$,
- $|w_k - w_{k+1}| \leq \epsilon/2$, and
- $B(w_k; \epsilon)$ lies in U.

This completes the proof. □

Solution 13.12. Suppose that f (say) vanishes to infinite order at z_0. By Theorem 4.13 the series coefficients and the derivatives determine each other. So, taking derivatives and applying the product rule, we find that fg also vanishes to infinite order at z_0.

Suppose now that both $\text{Ord}(f, z_0)$ and $\text{Ord}(g, z_0)$ are finite. By Lemma 13.11 we can factor the corresponding powers of $(z - z_0)$ out of f and g. Exponents add when multiplying powers, so we are done. □

Solution 14.3. Note that Z is closed in U, so $U - Z$ is open. For every $z_k \in Z$ we have some r_k so that $B_k = B(z_k; r_k) \subset U$ and also $B_k \cap Z = \{z_k\}$. We deduce that $U - Z$, together with the balls B_k, cover U. Thus $U - Z$, and the balls B_k, cover K. So there is a finite subcover. This finite subcover in particular covers $Z \cap K$. But $U - Z$ covers no point of Z and each B_k covers at most one point of $U - Z$. Thus $Z \cap K$ is finite. $\square$

Solution 14.4. The first claim follows from the definition.

We prove that contrapositive of the second claim. Suppose that $w \in U$ is an accumulation point of $Z \cup Z'$. So there is some sequence $(z_n)_n \subset Z \cup Z'$ so that, for all n, we have

- $z_n \neq w$ and
- $z_n \in B(w; 1/n)$.

By the pigeonhole principle one of Z or Z' contains infinitely many of the z_n. Say Z, as the other case is similar. Thus w is an accumulation point, in U, of Z. $\square$

Solution 14.5. We must show that $U - Z$ is non-empty, open, and connected.

If Z is empty we are done. Suppose that Z is not empty. Fix z_0 in Z. Suppose that $r_0 > 0$ has $B(z_0; r_0) \subset U$ and $B(z_0; r_0) \cap Z = \{z_0\}$. In particular $B(z_0; r_0) - \{z_0\}$ is contained in $U - Z$. So $U - Z$ is non-empty.

Since Z contains all of its accumulation points in U, it is closed in U. Thus $U - Z$ is open.

Finally, suppose that w_0 and w_1 are points of $U - Z$. Appealing to Lemma 7.3 we have a simple piecewise-linear contour $\gamma\colon [0, 1] \to U$ so that $\gamma(0) = w_0$ and $\gamma(1) = w_1$. Let C be the image of γ. Note that $Z \cap C$ has no accumulation point in C. Since C is compact, we deduce that $Z \cap C$ is finite. Pick $\epsilon > 0$ small as follows. For all $z_0 \in Z \cap C$ we have:

- $B(z_0; \epsilon) \subset U$,
- $B(z_0; \epsilon) \cap Z = \{z_0\}$,
- $B(z_0; \epsilon) \cap A$ is a pair of radii of $B(z_0; \epsilon)$. (We have to allow the possibility that z_0 is a breakpoint of γ.)

Recall that $C(z_0; \epsilon/3)$ is the circle centred at z_0 with radius $\epsilon/3$. As z_0 ranges over Z these are all disjoint from each other. We now "cut-and-paste", as follows. For every $z_0 \in Z \cap C$ we replace $A \cap B(z_0; \epsilon/3)$ by one of the two arcs of $C(z_0; \epsilon/3) - A$. This gives the desired (simple, piecewise-smooth) contour in $U - Z$ connecting w_0 to w_1. $\square$

Solution 15.5. We freely apply Theorem 4.13, Corollary 12.9, and Example 4.6 to obtain the following.

Suppose that L is the given Laurent series. Let g be the principal part of L; define f to be the sum of the non-negative terms of L. Thus f is holomorphic on $B(z_0; R)$. Define $h(z) = g(z_0 + 1/z)$. we deduce that h is holomorphic on $B(0; 1/r)$. Since $g(z) = h(1/(z - z_0))$ we deduce that g is holomorphic on $\mathbb{C} - \overline{B(z_0; r)}$.

Thus the partial sums of the bi-infinite series converge uniformly in the annulus $A(0; r, R)$. Furthermore, that they converge to the holomorphic function $f + g$. □

Solution 16.13. Suppose that f (say) vanishes to infinite order at z_0. By Theorem 4.13 the series coefficients and the derivatives determine each other. So, taking derivatives and applying the product rule, we find that fg also vanishes to infinite order at z_0.

Suppose now that both $\mathrm{ORD}(f, z_0)$ and $\mathrm{ORD}(g, z_0)$ are finite. By Lemma 13.11 we can factor the corresponding powers of $(z - z_0)$ out of f and g. Exponents add when multiplying powers, so we are done. □

Solution 19.2. There are many such homeomorphisms; what follows is just one possible example We define $f\colon [0, 1) \to [0, \infty)$ by $f(r) = r/(-r + 1)$. We define $g\colon [0, \infty) \to [0, 1)$ by $g(r) = r/(r + 1)$. Note that f and g are continuous and are inverses. Thus they are inverses.

We define $F\colon \mathbb{D} \to \mathbb{C}$ by $F(re^{i\theta}) = f(r)e^{i\theta}$. We define $G\colon \mathbb{C} \to \mathbb{D}$ by $G(re^{i\theta}) = g(r)e^{i\theta}$. These are continuous, and are inverses, because f and g are. □

Solution 19.3. Let $f\colon [a, b] \to [c, d]$ be the unique affine map taking a to c and b to d. Let $g\colon [c, d] \to [a, b]$ be the unique affine map taking c to a and d to b. That is:

$$f(s) = \frac{b - s}{b - a} \cdot c + \frac{s - a}{b - a} \cdot d g(t) = \frac{d - t}{d - c} \cdot a + \frac{s - c}{d - c} \cdot b$$

These are inverses and so are homeomorphisms of the intervals.

We define $F\colon A \to C$ by $F(re^{i\theta}) = f(r)e^{i\theta})$. Likewise, we define $G\colon C \to A$ by $G(re^{i\theta}) = g(r)e^{i\theta})$. These are continuous in continuous in polar coordinates, so are continuous. Also, since f and g are inverses, the same is true of F and G. □

Solution 19.5. We define $f\colon U \to C(\pi)$ by $f(z) = z^2$. We define $g\colon C(\pi) \to U$ by $g(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$. To prove that g is continuous requires the continuity of the argument; this was proved in Lemma 1.10. □

Solution 19.6. We define $f\colon A \to C$ by $f(re^{i\theta}) = (e^{i\theta}, r - 1)$. We define $g\colon C \to A$ by $g(e^{i\theta}, t) = (t + 1)e^{i\theta}$. These are continuous and inverse to each other. □

Solution 19.9. Fix a point w in U. Suppose that $\sigma^1 \in C_1(U)$ is a singular one-simplex. We define $d_w(\sigma^1) \in C_2(U)$ following Example 8.14. That is:

$$d_w(\sigma^1)(x_0, x_1, x_2) = \begin{cases} w, & \text{if } x_2 = 1 \\ x_2 w + (1 - x_2)\sigma^1\left(\frac{x_0}{1-x_2}, \frac{x_1}{1-x_2}\right) & \text{if } x_2 \neq 1 \end{cases}$$

Suppose now that $c = \sum \sigma_k^1$ is a one-cycle in U. We define $d = \sum d_w(\sigma_k^1)$. So this is a two-chain in U. If $\sigma_k^1|[v_1] = \sigma_j^1|[v_0]$ as zero-simplices, then $d_w(\sigma_k^1)|[v_1, v_2] = d_w(\sigma_j^1)|[v_0, v_2]$ as one-simplices. Also, $d_w(\sigma_k^1)|[v_0, v_1] = \sigma_k^1$, again as one-simplices. So $\partial d = \sum \partial d_w(\sigma_k^1) = \sum \sigma_k^1 = c$. So c is a one-boundary, as desired. □

Solution 19.10. Morally, this holds because homeomorphism "commute with the boundary operator". Here are some of the details.

Suppose that $f\colon U \to V$ and $g\colon V \to U$ are the given (inverse) homeomorphisms.

Suppose that $H_1(V) = 0$. Suppose that a is a one-cycle in U. So $c = f_*(a)$ is a one-cycle in V; this is the sum of the simplices of a postcomposed with f. So c is a one-boundary. Suppose that d is a two-chain in V with $\partial d = c$. Define $b = g_*(d)$; this is the sum of the simplices of d postcomposed with g. So b is a two-chain in U. We compute the homological boundary of b as follows.

$$\partial b = \partial(g_*(d)) = g_*(\partial d) = g_*(c) = g_*(f_*(a)) = (g \circ f)_*(a) = (\mathrm{Id}_U)_* a = a$$

Thus a is a one-boundary. Since a was arbitrary, we deduce that $H_1(U) = 0$. □

Solution 19.11. The plane $\mathbb{C}$ is convex, so by Exercise 19.9 has $H_1(\mathbb{C}) = 0$. On the other hand, by Example 19.8 the punctured plane $\mathbb{C}^\times$ has $H_1(\mathbb{C}^\times) \neq 0$. (Similarly for the punctured disc and the annulus.) We now appeal to Exercise 19.10. □

Solution 20.3. By Theorem 9.2 f integrates to zero along all closed contours. By Lemma 7.6 we find that f has a primitive. □

Solution 20.8. We parametrise $C(R)$ using $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, \pi]$. So $g(\gamma(\theta)) = \ln(R) + i\theta$ and $g^2(\gamma(\theta)) = \ln^2(R) + 2i\theta \ln(R) - \theta^2$. So, for R large enough, we have $|g^2(\gamma(\theta))| \leq 3\ln^2(R)$. So the magnitude of the integrand is dominated by a constant times $\ln^2(R)/R^2$. The claim (for $C(R)$) now follows from the ML–inequality (2.22).

To deal with $C(\epsilon)$ we use the substitution $z = -1/w$. This transforms the domain of integration into $C(1/\epsilon)$, oriented anti-clockwise. The integrand transforms as follows.

$$\begin{aligned} \frac{g^2(z)}{1+z^2}\,dz &= \frac{g^2(-w^{-1})}{1+w^{-2}}\,d(-w^{-1}) \\ &= \frac{(g(-1) - g(w))^2}{w^2+1}\,dw \\ &= \frac{(g(w) - \pi i)^2}{w^2+1}\,dw \end{aligned}$$

We now argue as in the previous paragraph. □

Solution 21.4. Suppose that $p(z) = \sum_{k=0}^{N} a_k z^k$ with $a_N \neq 0$. Define $A = (\sum_{k=0}^{N-1} |a_k|)/|a_N|$. Suppose that $|z| \geq \max 2A, 1$. Then we have

$$\begin{aligned}
|p(z)| &\geq |a_N||z^N| - (\sum_{k=0}^{N-1} |a_k||z^k|) && \text{triangle inequality} \\
&\geq |a_N||z^N| - (\sum_{k=0}^{N-1} |a_k|)|z|^{N-1} && |z| \geq 1 \\
&\geq |z|^{N-1}(|a_N||z| - (\sum_{k=0}^{N-1} |a_k|) && \text{algebra} \\
&\geq |z|^{N-1}(|a_N||z|/2 + |a_N|A - (\sum_{k=0}^{N-1} |a_k|) && |z| \geq 2A \\
&\geq |a_N||z|^N/2 && \text{definition of } A \\
&\geq |a_N|2^{N-1}A^N && |z| \geq 2A
\end{aligned}$$

Thus $\frac{1}{|p(z)|} \leq \frac{1}{|a_N|2^{N-1}A^N}$ is bounded near infinity, as desired. □

Solution 22.8. Note that U is non-empty, connected, and open. So V is non-empty. Suppose that c and d are points of V. Since $V = f(U)$ there are points a and b in U so that $f(a) = c$ and $f(b) = d$. Recall that U is path-connected (Exercise 2.4). Suppose that α is a path in U from a to b. So $\beta = f \circ \alpha$ is a path from c to d. Thus V is path-connected. Finally, V is open by the open mapping theorem (22.7). □

Solution 23.10. We take $f(z) = \frac{z-u}{v-u}$. Suppose that g is another similarity sending u and v to 0 and 1 (in order). Define $h = f \circ g^{-1}$. Thus $h(z) = az + b$ is a similarity fixing zero and one. So $b = 0$ and $a = 1$; thus $h = \mathrm{Id}_{\mathbb{C}}$. □

Solution 24.2. Stereographic projection and its inverse extend to $\hat{\mathbb{C}}$ to give the desired homeomorphisms. □

Solution 24.3. The closure $\hat{L}$ is one-point compactification of L. Thus the closure is homeomorphic to a circle. □

Solution 24.4. Note that A_{-w}, $M_{1/w}$, and V are inverses for A_w, M_w, and V, respectively. So we only need to check continuity near infinity (and near zero, for V). This follows for A_w and M_w because they send compact subsets of the plane to compact subsets. It follows for V because it interchanges neighbourhoods of zero and infinity. □

Solution 24.5. The inversion V sends:

- the real axis to itself,
- the imaginary axis to itself,
- the unit circle to itself (by reflection),
- circles centred on the origin to circles centred on the origin,

- lines meeting the origin to lines meeting the origin,
- lines not meeting the origin to circles meeting the origin,
- circles meeting the origin to lines not meeting the origin,
- circles not meeting the origin to circles not meeting the origin, and
- circles and lines perpendicular to the unit circle to circles and lines perpendicular to the unit circle.

We can find algebraic proofs of these by, say, transforming defining equations of circles and lines in various ways. More geometric proofs can be deduced from the identification of the extended plane and the round two-sphere via *stereographic projection*. □

Solution 24.7. We first assume that neither L nor M meets the origin. Thus $L' = V(\hat{L})$ and $M' = V(\hat{M})$. Also, by Exercise 24.5, the images L' and M' are circles through the origin. Since L and M are parallel, they do not meet. Thus $\hat{L}$ and $\hat{M}$ meet only at infinity. Thus L' and M' meet only at the origin. Thus they are tangent at the origin.

We can now discuss the "combinatorics" of L' and M'.

- If L separates the origin from M then M' is contained inside the disc bounded by L'.
- If M separates the origin from L then L' is contained inside the disc bounded by M'.
- If the origin is between L and M then neither circle is contained inside the disc bounded by the other.

Suppose now that L meets the origin. Thus M does not meet the origin. By Exercise 24.5 we have that L' is again a line, while M' is a circle. These meet only at the origin, thus they are tangent. If M meets the origin then L does not and the roles are reversed. □

Solution 24.13. Let f and g by as given in the theorem. Then f has inverse $z \mapsto z/a$ while g is its own inverse. So f and g are biholomorphic and thus are automorphisms.

Suppose that $h\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is an automorphism. We apply Theorem 15.6 to find a Laurent series for h converging uniformly on compact subsets of $\mathbb{C}^\times$. We now argue as in the proof of Theorem 23.9. That is, if h has an essential singularity (at zero or at infinity) then it is not injective. This is a contradiction; we deduce that the Laurent series for h has finitely many terms.

If the series has at least two terms, then clearing denominators and applying Theorem 21.3 we find that h has a root, a contradiction. We deduce that the series has only one term. If the degree of the term is zero then h is constant, a contradiction. If the degree is less than minus one, or greater than one, then h is not injective, a contradiction. So there is only one term and this term has degree minus one or plus one. Thus there is some $a \in \mathbb{C}^\times$ so that $h(z) = az$ or $h(z) = a/z$. □

Solution 25.4. Suppose that $A \in \mathrm{SL}(2,\mathbb{C})$ is a matrix. Suppose that $[A]_S$ and $[A]_G$ are its equivalence classes in $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$, respectively. Note that $[A]_S \subset [A]_G$; thus $\rho_{\mathbb{C}}([A]_S) = [A]_G$ and so $\rho_{\mathbb{C}}$ is well-defined. Next, suppose that $[A]_S$ and $[B]_S$ are two classes in $\mathrm{PSL}(2,\mathbb{C})$. Then $[AB]_S$ is a subset of $[AB]_G$. So $\rho_{\mathbb{C}}$ is a homomorphism. The proof is the same replacing $\mathbb{C}$ by $\mathbb{R}$.

Suppose that A is any matrix in $GL(2,\mathbb{C})$. Let $\sqrt{\det(A)}$ be either square root of the determinant. Let $D = \frac{1}{\sqrt{\det(A)}}\,\mathrm{Id}$. Let $A' = A \cdot D$. So A' lies in $\mathrm{SL}(2,\mathbb{C})$. Thus $\rho_{\mathbb{C}}([A']_S) = [A]_G$ and so $\rho_{\mathbb{C}}$ is surjective.

Suppose that A is any matrix in $GL(2,\mathbb{R})$ with negative determinant. Then $[A]_G$ is not in the image of $\rho_{\mathbb{R}}$. □

Solution 25.5. Suppose that $\lambda \in \mathbb{C}^\times$. Take $B = \lambda \cdot A$. So $[A]_G$ and $[B]_G$ are equal in $\mathrm{PGL}(2,\mathbb{C})$. Cancelling copies of λ, we find that $f_A = f_B$. Thus ρ is well-defined.

The function ρ is surjective by Lemma 24.10. The remainder of the exercise can be done algebraically.

Another way to package the remaining work is to "notice" that $\hat{\mathbb{C}}$ is isomorphic (as a Riemann surface) to $\mathbb{CP}^1$, the complex projective line. Now, $\mathbb{CP}^1 = (\mathbb{C}^2 - \{0\})/(\mathbb{C} - \{0\})$, so the usual action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2$ descends. □

Solution 25.8. Suppose that A and B lie in $\mathrm{U}(1,1)$. Note that $\det(A^*) = \overline{\det(A)}$. Taking the determinant of $A^*JA = J$ we find that $|\det(A)|^2 = 1$. So the determinant is non-zero; so A is invertable. Note that $(A^{-1})^* = (A^*)^{-1}$. So A^{-1} lies in $\mathrm{U}(1,1)$.

So $(AB)^*JAB = B^*A^*JAB = B^*JB = J$. Thus AB again lies in $\mathrm{U}(1,1)$.

The proof for signature $(2,0)$ is similar. □

Solution 25.9. The proofs are similar to the one for $\mathrm{PSL}(2,\mathbb{C})$. □

Solution 25.10. Suppose that A lies in $\mathrm{U}(1,1)$; suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The defining equation $A^*JA = J$ implies

$$\begin{aligned} a\bar{b} &= c\bar{d} \\ \bar{a}b &= \bar{c}d \\ |a|^2 - |c|^2 &= 1 \\ |d|^2 - |b|^2 &= 1 \end{aligned}$$

Taking the determinant of $A^*JA = J$ and simplifying also gives

$$\begin{aligned} |a|^2 - |b|^2 &= 1 \\ |d|^2 - |c|^2 &= 1 \end{aligned}$$

Suppose now that A lies in $\mathrm{SU}(1,1)$. So we additionally have $ad-bc=1$. Multiplying this by $\bar{a}$ and simplifying implies $d=\bar{a}$. We similarly find that $c=\bar{b}$, so we are done.

The proof for $A \in \mathrm{SU}(2)$ is similar. □

Solution 25.12. We first suppose that none of u, v, and w are the point at infinity. Then we define:

$$f(z) = \frac{w-u}{w-v} \cdot \frac{z-v}{z-u}$$

Note that f sends u, v, and w to ∞, 0, and 1 (in order).

There are three special cases as u, v, or w is the point at infinity. In these cases, respectively, we take $f(z)$ to be the following:

$$\frac{z-v}{w-v} \qquad \frac{w-u}{z-u} \qquad \frac{z-v}{z-u}$$

Again, f sends u, v, and w to ∞, 0, and 1 (in order).

Suppose that g is another such linear fractional transformation. Then $h = f \circ g^{-1}$ fixes ∞, 0, and 1. Since h fixes ∞ we deduce that h is (the extension of) a similarity (Exercise 25.14). Since h fixes 0 and 1 we deduce that h is the identity (Exercise 23.10). □

Solution 25.14. Suppose that a, b, c, and d lie in $\mathbb{C}$, with $ad-bc \neq 0$. Suppose that $f(z) = (az+b)/(cz+d)$ fixes infinity. We deduce that $1/f(1/z) = (dz+c)/(bz+a)$ fixes zero. Thus $c = 0$. So $f(z) = (a/d)z + (c/d)$ is a similarity, as desired. □

Solution 25.15. Suppose that $f \in \mathrm{Aut}(\hat{\mathbb{C}})$ preserves $\hat{\mathbb{R}}$ setwise. Suppose that u, v, and w are the preimages of 0, ∞, and 1 (in that order) under f. Thus u, v, and w lie in $\hat{\mathbb{R}}$. Suppose that none of u, v, or w equal ∞. By Exercise 25.12 we have

$$f(z) = \frac{w-u}{w-v} \cdot \frac{z-v}{z-u}$$

(If one of u, v, or w is ∞, then we replace f by the corresponding transformation given in Exercise 25.12.) So f lies in $\rho(\mathrm{PGL}(2,\mathbb{R}))$, as desired. There is no kernel by the uniqueness proved in Exercise 25.12.

The action of $\mathrm{PGL}(2,\mathbb{R})$ is three-transitive as before (the proof is the same and part of it appears in the previous paragraph).

Consider the triple $(0,\infty,-1)$. The unique linear fractional transformation sending these to $(0,\infty,1)$, in that order, is $f(z) = -z$. This lies in the image of $\mathrm{PGL}(2,\mathbb{R})$, but not in the image of $\mathrm{PSL}(2,\mathbb{R})$. So the action of $\mathrm{PSL}(2,\mathbb{R})$ is not three-transitive.

Suppose that (u,v,w) is any anticlockwise triple in $\hat{\mathbb{R}}$. Suppose that none of u, v, or w are the point at infinity. We must show that the transformation

$$f(z) = \frac{w-u}{w-v} \cdot \frac{z-v}{z-u}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This is equivalent to showing that the determinant

$$\frac{w-u}{w-v}(-u) - \frac{w-u}{w-v}(-v) = \frac{(w-u)(v-u)}{w-v}$$

is positive. Since (u, v, w) are not infinite, and are ordered anticlockwise in $\hat{\mathbb{R}}$, we have either

$$\begin{aligned} u < v < w&, \\ v < w < u&, \quad \text{or} \\ w < u < v&. \end{aligned}$$

We observe that the determinant is positive in all three cases, so are done.

Suppose that u is the point at infinity. So we must show that

$$f(z) = \frac{z-v}{w-v}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $w - v$ is positive, if and only $v < w$, if and only if (u, v, w) is ordered anticlockwise.

Suppose that v is the point at infinity. So we must show that

$$f(z) = \frac{w-u}{z-u}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $u - w$ is positive, if and only $w < u$, if and only if (v, w, u) is ordered anticlockwise, if and only if (u, v, w) is ordered anticlockwise.

Suppose that w is the point at infinity. So we must show that

$$f(z) = \frac{z-v}{z-u}$$

lies in $\rho(\mathrm{PSL}(2,\mathbb{R}))$. This happens if and only if $v - u$ is positive, if and only $u < v$, if and only if (w, u, v) is ordered anticlockwise, if and only if (u, v, w) is ordered anticlockwise.

Suppose that $f \in \mathrm{A{\scriptsize UT}}(\hat{\mathbb{C}})$ lies in $\mathrm{PSL}(2,\mathbb{R})$. We must prove that f preserves $\mathbb{H}$ setwise. Since f preserves $\hat{\mathbb{R}}$ we deduce that f either preserves $\mathbb{H}$ or sends it to the lower half plane. So take $f(z) = (az + b)/(cz + d)$ and compute as follows:

$$\begin{aligned} f(i) &= \frac{ai+b}{ci+d} \\ &= \frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)} \\ &= \frac{ac+bd+i(ad-bc)}{c^2+d^2} \\ &= \frac{ac+bd+i}{c^2+d^2} \end{aligned}$$

So the imaginary part of $f(i)$ is $1/(c^2 + d^2)$. This is positive, so f preserves $\mathbb{H}$ setwise, as desired. $\square$

Solution 25.16. To find the automorphism f, we send the triple $(i, -i, 1)$ to the triple $(\infty, 0, 1)$. This gives:

$$f(z) = \frac{1-i}{1+i} \cdot \frac{z+i}{z-i} = \frac{1}{i} \cdot \frac{z+i}{z-i} = \frac{z+i}{iz+1}$$

(There are many other ways to proceed. For example, consider the linear fractional transformation that sends the triple $(0, 1, i)$ to the triple $(i, 1, \infty)$.) Fix any $z \neq i$. We take $z = re^{i\theta}$ and compute as follows.

$$\begin{aligned} f(z) &= \frac{z+i}{iz+1} \\ &= \frac{(z+i)(-i\bar{z}+1)}{|iz+1|^2} \\ &= \frac{-i|z|^2 + z + \bar{z} + i}{|iz+1|^2} \\ &= \frac{(z+\bar{z}) + i(1-|z|^2)}{|iz+1|^2} \\ &= \frac{2r\cos(\theta) + i(1-r^2)}{|iz+1|^2} \end{aligned}$$

So, $r < 1$ if and only if $\textsc{Imag}(f(z)) > 0$. Also, $r = 1$ if and only if $\textsc{Imag}(f(z)) = 0$. Since f is an automorphism, it has an inverse and so is a bijection. We deduce that $f(\mathbb{D}) = \mathbb{H}$ and $f(S^1) = \hat{\mathbb{R}}$.

The remainder of the proof follows from the definition of the stabiliser $\textsc{Aut}_X(\hat{\mathbb{C}})$. $\square$

Solution 25.17. By the previous exercise

$$f(z) = \frac{z+i}{iz+1}$$

sends S^1 to $\hat{\mathbb{R}}$. Note also that $f(0) = i$. So f sends $\mathbb{D}$ to $\mathbb{H}$. We deduce that f conjugates $\textsc{Aut}_{\mathbb{H}}(\hat{\mathbb{C}})$ to $\textsc{Aut}_{\mathbb{D}}(\hat{\mathbb{C}})$. We now take

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Note that A lies in $\mathrm{PSL}(2, \mathbb{C})$ and that $\rho([A]) = f_A = f$.

We claim that A conjugates $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSU}(1, 1)$. We prove this in two steps. First suppose that a, b, c, and d are real and that $ad - bc = 1$. We take

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We now compute as follows:

$$\begin{aligned}
A^{-1}BA &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} a - ic & b - id \\ -ia + c & -ib + d \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} a - ic + ib + d & ia + c + b - id \\ -ia + c + b + id & a + ic - ib + d \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (a + d) + i(b - c) & (c + b) + i(a - d) \\ (c + b) - i(a - d) & (a + d) - i(b - c) \end{pmatrix}
\end{aligned}$$

The final matrix lies in $\mathrm{SU}(1,1)$. Thus its image under ρ lies in $\mathrm{PSU}(1,1)$.

Second suppose that p and q are complex numbers with $|p|^2 - |q|^2 = 1$. We take

$$C = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}$$

We now compute as follows:

$$\begin{aligned}
ACA^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} p + i\bar{q} & q + i\bar{p} \\ ip + \bar{q} & iq + \bar{p} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} p + i\bar{q} - iq + \bar{p} & -ip + \bar{q} + q + i\bar{p} \\ ip + \bar{q} + q - i\bar{p} & p - i\bar{q} + iq + \bar{p} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (p + \bar{p}) - i(q - \bar{q}) & -i(p - \bar{p}) + (q + \bar{q}) \\ i(p - i\bar{p}) + (q + \bar{q}) & (p + \bar{p}) + i(q - \bar{q}) \end{pmatrix} \\
&= \begin{pmatrix} \textsc{Real}(p) + \textsc{Imag}(q) & \textsc{Imag}(p) + \textsc{Real}(q) \\ -\textsc{Imag}(p) + \textsc{Real}(q) & \textsc{Real}(p) - \textsc{Imag}(q) \end{pmatrix}
\end{aligned}$$

The final matrix lies in $\mathrm{SL}(2,\mathbb{R})$. Thus its image under ρ lies in $\mathrm{PSL}(2,\mathbb{R})$. Thus conjugation by A, restricted to $\mathrm{PSL}(2,\mathbb{R})$, gives an isomorphism from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PSU}(1,1)$.

The linear fractional transformation $f = f_A$ allows us to "translate" back and forth between the stabilisers of the circle and disc, on the one hand, and the stabilisers of the extended line and the upper half plane, on the other hand. Similarly, the matrix A allows us to "translate" between the groups $\mathrm{PGL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$, on the one hand, and the groups $\mathrm{PU}(1,1)$ and $\mathrm{PSU}(1,1)$, on the other hand. Finally, we complete the picture by noting that ρ takes the various matrix groups to the various stabilisers. □

Index of symbols

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