2025-02-18 SAUL SCHLEIMER MA4H4 LECTURE 19

① OUTCENTRES

WE CAN SHARPEN THE BOUND ON TORSION IN HYP. GROUPS. HERE IS THE NECESSARY TOOL.

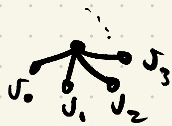
DEF: SUPPOSE (X, d_X) IS A METRIC SPACE. SUPPOSE $\Delta \subset X$ FINITE. THE *OUTRADIUS* OF Δ IS $\inf \left\{ r \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{THERE IS } x \in X \\ \text{WITH } \Delta \subset B_X(x, R) \end{array} \right\}$

WE CALL $x \in X$ AN *OUTCENTRE* FOR Δ IF $x \in X$ *REALISES* THE OUTRADIUS. LET $\mathrm{CEN}(\Delta)$ BE THE SET OF OUTCENTRES.

DEF: SUPPOSE (X, d_X) IS A METRIC SPACE. WE SAY X IS *PROPER* IF ALL METRIC BALLS $B_X(x, R)$ ARE COMPACT.

EXAMPLE: LOCAL FINITE GRAPHS.

NON-EXAMPLE. LOCALLY INFINITE GRAPHS.



EXERCISE: IF (X, d_X) IS PROPER THEN IT IS COMPLETE. (THAT IS, CAUCHY SEQS CONVERGE).

EXERCISE: SUPPOSE (X, d_X) IS PROPER. SUPPOSE Δ IS FINITE AND NONEMPTY. THEN $\mathrm{CEN}(\Delta)$ IS NONEMPTY.

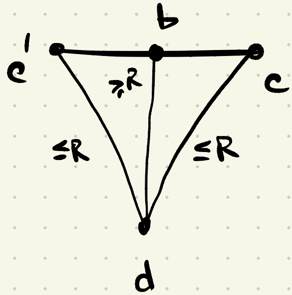
EXERCISE: PROVE THAT $\mathrm{CEN}(\Delta)$ CAN HAVE LARGE DIAM IN $\mathbb{Z}^2$. (UNLIKE $\mathbb{E}^2$!)

LEMMA: SUPPOSE (X, d_X) IS δ-HYPERBOLIC AND PROPER. SUPPOSE $\Delta \subset X$ IS FINITE, NONEMPTY. THEN

$$\mathrm{DIAM}(\mathrm{CEN}(\Delta)) \leq 4\delta.$$

PROOF: SUPPOSE $c, c' \in \mathrm{CEN}(\Delta)$. LET γ BE A GEODESIC. LET $b \in \gamma$ BE THE MIDPOINT. PICK $d \in \Delta$ TO MAXIMISE $d_X(b, d)$. SO $d_X(c, d), d_X(c', d) \leq R \leq d_X(b, d)$.

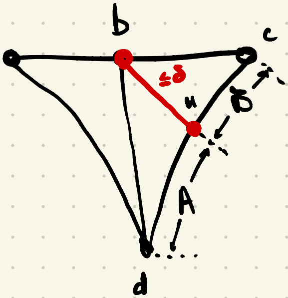
PICTURE:



PICK GEODESICS β AND β' FROM d TO c AND c'.

AS THE OTHER CASE IS SIMILAR SUPPOSE $u \in \beta$ HAS $d_X(b,u) \leq \delta$.

PICTURE



SUPPOSE $d_X(d,u) = A$, $d_X(u,c) = B$.

SO
$$\begin{aligned} A + B &\leq d_X(c,d) \\ &\leq R \\ &\leq d_X(b,d) \\ &\leq A + \delta \end{aligned}$$

CHAR. δ-HYP?

SO, AGAIN, $B \leq \delta$ AND SO $d_X(c,c') \leq 4\delta$. $\square$

② MUCH BETTER BOUNDS ON TORSION:

THEOREM [BRADY 2000 [AND OTHERS]] SUPPOSE (G,S) IS δ-HYPERBOLIC. SUPPOSE $F < G$ IS FINITE. THEN THERE IS SOME $h \in G$ SO THAT $h^{-1}Fh \subset B(4\delta + 1)$.

PROOF: PICK $h \in G$ WHICH IS AS CLOSE AS POSSIBLE TO $\mathrm{CEN}(F)$. SO h LIES WITHIN DIST $\leq 1/2$ of SOME CENTRE $c \in \mathrm{CEN}(F)$. THUS $d_S(h^{-1}c, 1_G) \leq 1/2$.

CLAIM: $h^{-1} \cdot \mathrm{CEN}(F) = \mathrm{CEN}(h^{-1} \cdot F)$ [NOTE $\mathrm{DIAM}(F) = \mathrm{DIAM}(h^{-1}F)$ MAY BE LARGE!]

PROOF: G ACTS ON $\Gamma(G,S)$ BY ISOMETRIES. $\square$

CLAIM: $h^{-1}Fh$ PEMUTES THE ELEMENTS of $\mathrm{CEN}(h^{-1}F)$.

PROOF: F PERMUTES THE ELEMENTS of $\mathrm{CEN}(F)$. $\square$

CLAIM: $h^{-1}Fh \subset B_S(1_G, 4\delta + 1)$.

PROOF: $h^{-1}c$ IS CLOSE TO 1_G. APPLY LEMMA $\square$

THIS PROVES THE THEOREM. □

③ THE BEGINNING of QUASI-GEODESICS

PROPOSITION (VER 2): SUPPOSE X IS A δ-HYPERBOLIC METRIC SPACE. FIX $k > 4\delta$ (AND EVEN, SAY). SUPPOSE $\alpha : [0,n] \longrightarrow X$ IS A k-LOCAL GEODESIC WITH $\alpha(0)$ A VERTEX.

THEN:

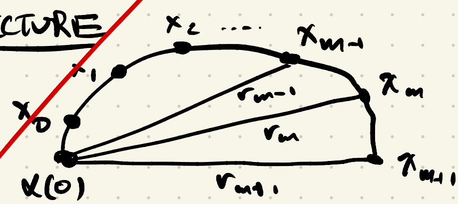
$$\frac{n}{k/2} \leq d_X(\alpha(0), \alpha(n)) \leq n.$$

NOT DONE.

PROOF: SINCE $k > 0$ WE DEDUCE α IS AN EDGE PATH. THIS GIVES THE UPPER BOUND. TO PROVE THE LOWER BOUND:

SET
$$\varepsilon = n \bmod k/2$$
$$x_m = \varepsilon + m \cdot k/2$$
$$r_m = d_X(\alpha(0), \alpha(x_m))$$

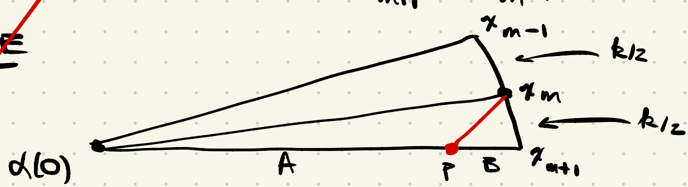
PICTURE



SO IT SUFFICES TO PROVE $r_{m+1} \geq r_m + 1$. NOTE THAT $r_m \in \mathbb{N}$ SO IT SUFFICES TO PROVE (r_m) IS STRICTLY INCREASING.

EXERCISE: $0 < r_0 < r_1$.

SUPPOSE (FOR CONTRADICTION) THAT r_m IS NOT STRICTLY INCREASING. FIX MINIMAL m SO THAT $r_{m+1} \leq r_m$. NOTE $r_{m-1} < r_m$. PICTURE



LET β_m BE A GEODESIC FROM $\alpha(0)$ TO x_m. SO $\alpha|[x_{m-1}, x_{m+1}]$, β_{m-1}, β_{m+1} FORM A GEODESIC TRIANGLE. SO x_m IS WITHIN δ of SOME POINT p of β_{m+1} (SAY). WE FIND $k/2 \leq 2\delta$, GIVING A CONTRADICTION. □