2025-02-20 MA444 LECTURE 21 SAUL SCHLEIMER

① DEHN PRESENTATIONS

THEOREM: SUPPOSE (G,S) IS δ-HYPERBOLIC. SET
$$R = \{ w \in F(S) \mid w =_G 1_G \text{ AND } |w| \le 8\delta + 1 \}.$$
THEN $\langle S \mid R \rangle$ IS A DEHN PRESENTION.

PROOF: SUPPOSE $w \in F(S)$ HAS $w =_G 1_G$. SO w IS A LOOP, SO NOT $(4\delta+1)$-LOCAL GEODESIC. [PROP VER 1] SO WE CAN FACTOR $w = u p v$ AS BEFORE AND REPLACE p BY q USING $pq^{-1} \in R$, AS ABOVE $\square$.

THEOREM: WORD HYP GROUPS HAVE SOLVABLE WORD PROBLEM. THAT IS: IF (G,S) IS A δ-HYP GROUP THEN THERE EXISTS AN ALGORITHM WHICH, GIVEN $w \in F(S)$, DECIDES IF $w =_G 1_G$.

REMARK: THIS IS AN EXISTENCE RESULT, BUT NOT A CONSTRUCTIVE ONE! KNOWING (G,S) IS NOT ENOUGH TO TELL US δ, OR $\Gamma(G,S)$, ETC.

PROOF OF THEOREM: GIVEN $w \in F(S)$, THE ALGORITHM SCANS IT LOOKING FOR NON-GEOD. WORDS p OF LEN. $\le 4\delta+1$. IF IT FINDS ANY, IT "SHORTENS": REPLACES p BY A GEOD WORD $q =_G p$. THE ALGORIHM CONTINUES UNTIL w HAS NO REDUCTIONS.

NOW, IF $w = \varepsilon_S$ PRINT "YES" AND HALT

IF $w \ne \varepsilon_S$ PRINT "NO" AND HALT

EXERCISE: PROVE TERMINATION AND CORRECTNESS. ESTIMATE RUNNING TIME.

EXERCISE: SOLVE THE WORD PROBLEM FOR $\mathbb{Z}^D$ (IN LINEAR TIME).

② LINEAR ISOPERIMETRIC FUNCTION

THM: SUPPOSE (G,S) GIVEN. THE FOLLOWING ARE EQUIV.

(i) (G,S) IS δ-HYP

(ii) (G,S) HAS A DEHN PRES.

(iii) (G,S) HAS ~~LINEAR ISOPER INEQU.~~

(iv) (G,S) HAS ~~SUB-QUAD ISOPER. INEQU.~~

SKIP.

WE OMIT THE PROOF.

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③ THE CONJUGACY PROBLEM

DEF: SUPPOSE (G,S) IS A FIN GEN GROUP. THE CONJ. PROBLEM FOR (G,S) ASKS FOR AN ALGORITHM WHICH, GIVEN $u, v \in F(S)$, DECIDES IF THERE IS SOME $w \in F(S)$ SO THAT $uw =_G wv$.

THEOREM: SUPPOSE (G,S) δ-HYPERBOLIC. THEN THERE EXISTS AN ALGO. TO SOLVE THE CONJ. PROB. FOR (G,S).

WE REQUIRE THE FOLLOWING LEMMA: [BH, PAGE 452]

LEMMA: SUPPOSE (G,S) δ-HYP. SUPPOSE $u, v \in F(S)$ ARE CONJ. IN G. SUPPOSE ALL ROTATIONS of u, v ARE GEODESIC WORDS IN $\Gamma(G,S)$.

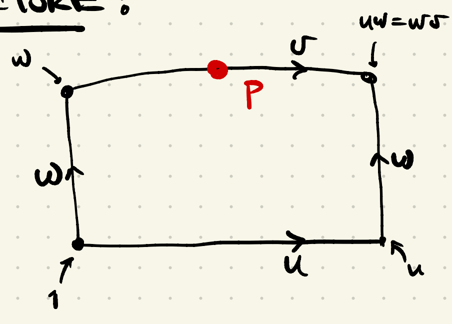
THEN EITHER (i) $|u|_S, |v|_S \leq 8\delta$

OR (ii) THERE ARE ROTATIONS u', v' of u, v AND $w \in F(S)$ SO THAT $|w|_S \leq 2\delta$ AND $u' \cdot w = w \cdot v'$.

PROOF: BUILD THE QUADRILATERAL Q AS SHOWN: LET p BE THE MIDPOINT of THE v-SIDE. [SO p A VERTEX OR AN EDGE MIDPOINT].

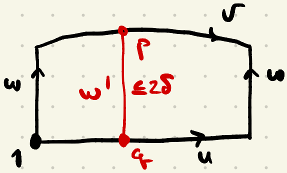
SUPPOSE $d_S(p, u\text{-SIDE}) \leq 2\delta$.

PICTURE:

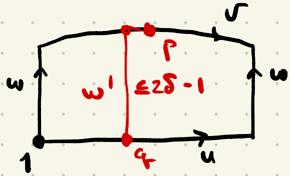


PICTURE

IF p VERTEX:

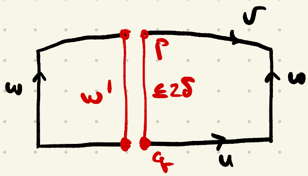


IF p MIDPT:

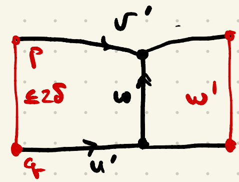


WE CUT AND PASTE AS FOLLOWS:

CUT



PASTE:



THIS GIVES CONCLUSION (ii).

SO SUPPOSE INSTEAD WE HAVE $|w| > 2\delta$ FOR ANY ROTATIONS u', v'!

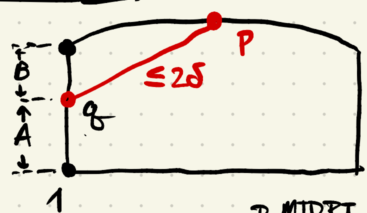
BY CUTTING/PASTING WE MAY ASSUME THAT THE w-SIDES ARE THE CLOSEST APPROACH BETWEEN VERTICES of THE u- AND v-SIDES.

SO $d_S(p, u\text{-SIDE}) \geq |w|_S \; (> 2\delta)$.

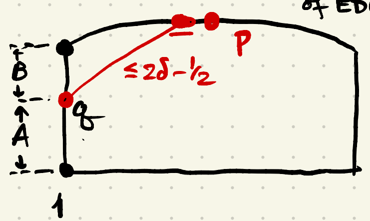
REFLECTING IF NEEDED, WE HAVE SOME q IN THE FIRST w-SIDE WITH $d_S(p,q) \leq 2\delta$.

PICTURES:

p VERTEX



p MIDPT of EDGE.



SO $A+B = |w|$ — w-SIDE

$\leq d_S(p, u\text{-SIDE})$ — CUT/PASTE HYPOTH.

$\leq d_S(p, 1)$ — 1_G IN u-SIDE

$\leq A + 2\delta$ — Δ-INEQU

SO $B \leq 2\delta$. THUS $d_S(p, w) \leq 4\delta$.

SINCE p IS THE MIDPOINT, $|v| \leq 8\delta$.

A SIMILAR ARGUMENT BOUNDS $|u|$. ∎