2025-02-25 LECTURE 22 SAUL SCHLEIMER MA4H4

## ① CONJUGACY PROBLEM

LAST TIME PROVED:

**LEMMA**: SUPPOSE $(G,S)$ $\delta$-HYP. SUPPOSE $u,v \in F(S)$ ARE CONJ. IN $G$. SUPPOSE ALL ROTATIONS of $u,v$ ARE GEODESIC WORDS IN $\Gamma(G,S)$.

**THEN** EITHER (i) $|u|_S, |v|_S \le 8\delta$

OR (ii) THERE ARE ROTATIONS $u', v'$ of $u, v$ AND $w \in F(S)$ SO THAT $|w|_S \le 2\delta$ AND $u' \cdot w = w \cdot v'$.

**COROLLARY**: THERE IS A CONSTANT $K = K(|S|, \delta)$ AS FOLLOWS:

SUPPOSE $(G,S)$ IS $\delta$-HYP. SUPPOSE $u, v \in G$ CONJUGATE.

THEN THERE IS $w \in G$ SO THAT

(i) $u \cdot w = w \cdot v$ AND

(ii) $|w|_S \le |u|_S + |v|_S + K$.

**EXERCISE**: PROVE THIS.

**EXERCISE**: SOLVE THE CONJUGACY PROBLEM FOR $(G,S)$ $\delta$-HYPERBOLIC.

[AGAIN: THE PROOF IS NON-CONSTRUCTIVE.]

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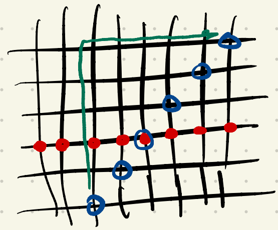
## ② QUASI-CONVEXITY.

**DEF**: SUPPOSE $(X, d_X)$ IS A GEODESIC METRIC SPACE. SUPPOSE $Y \subset X$ IS A SUBSET. WE SAY $Y$ IS $K$-QUASI-CONVEX IF, FOR ALL $y, y' \in Y$, FOR ALL GEODESICS $[y,y']$ WE HAVE $[y,y'] \subset N_X(Y, K)$.

**DEF**: SUPPOSE $(G,S)$ FIN GEN. SUPPOSE $H < G$ SUBGROUP. SAY $H < G$ IS QUASI-CONVEX IF THERE IS SOME $K$ SO THAT $H \subset \Gamma(G,S)$ IS $K$-QUASI-CONVEX.

EXAMPLE: TAKE $\Gamma = \Gamma(\mathbb{Z}^2, \text{STD})$. THEN THE SUBGROUP $\{(n,0) \mid n \in \mathbb{Z}\}$ IS QUASI-CONVEX ($K=1$) WHILE THE SUBGROUP $\{(n,n) \mid n \in \mathbb{Z}\}$ IS NOT QUASI-CONVEX (FOR ANY $K$). PICTURE



DEF: CALL $Y \subset X$ CONVEX IF IT IS $K$-QUASI-CONVEX FOR $K=0$.

LEMMA: SUPPOSE $H < G$ IS QUASI-CONVEX

THEN ① $H$ IS FIN GEN

AND ② $H$ IS UNDISTORTED.

PROOF: SUPPOSE $H \subset \Gamma(G,S)$ IS $K$-QUASI CONVEX. SET

$T = H \cap B_S(1_G, 2K+1)$. THEN $T$ IS FINITE. SUPPOSE $h \in H$.

FIX $\alpha : [0,n] \longrightarrow \Gamma(G,S)$ A GEODESIC FROM $1_G = 1_H$ TO $h$.

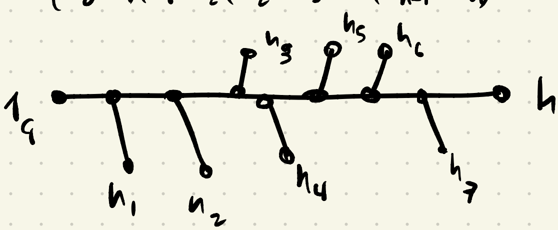
SO $\alpha(i) \in N_S(H,K)$. SO FIX $h_i \in H$ WITH $d_S(\alpha(i), h_i) \leq K$.

ALSO, PICK $h_0 = 1_G$, $h_n = h$. SO $d_S(h_i, h_{i+1}) \leq 2K+1$ AND SO

$d_S(1_G, h_i^{-1} h_{i+1}) \leq 2K+1$. THUS $h_i^{-1} h_{i+1} \in T$.

$$\text{ALSO: } (h_0^{-1} h_1)(h_1^{-1} h_2)(h_2^{-1} h_3) \cdots (h_{n-1}^{-1} h_n) = h_0^{-1} h_n = 1_G^{-1} h = h \qquad (1)$$

PICTURE




TO PROVE $H < G$ IS UNDISTORTED, WE NEED SOME CONST $C$

SO THAT $d_T(1_G, h) \leq C \cdot d_S(1_G, h)$ [NO "SHORTCUTS" USING $S$]

BUT $C=1$ WORKS (BECAUSE $T$ IS VERY LARGE!) $\square$

EXAMPLE: $\{(n,0)\} < \mathbb{Z}^2$ CONVEX $\Rightarrow$ QUASI CONVEX $\Rightarrow$ UNDISTORTED.

$\{(n,n)\} < \mathbb{Z}^2$ NOT QUASI CONVEX, BUT STILL UNDISTORTED.

$\langle b \rangle < \langle a, b \mid aba^{-1} = b^2 \rangle$ DISTORTED (SO NOT QC).

**CHALLENGE**: SUPPOSE $(G,S)$ $\delta$-HYPERBOLIC. SUPPOSE $H < G$.

**THEN** $H$ IS QUASI-CONVEX IFF

$H$ FIN GEN AND UNDISTORTED.