2025-02-26 LECTURE 23 SAUL SCHLEIMER MA4H4

① QUASI-CONVEX

EXAMPLE: $\{(n,0)\} < \mathbb{Z}^2$ CONVEX $\Rightarrow$ QUASI CONVEX $\Rightarrow$ UNDISTORTED.

$\{(n,n)\} < \mathbb{Z}^2$ NOT QUASI CONVEX, BUT STILL UNDISTORTED.

$\langle b \rangle < \langle a,b \mid aba^{-1} = b^2 \rangle$ DISTORTED SO NOT QC.

CHALLENGE: SUPPOSE (G,S) δ-HYPERBOLIC. SUPPOSE $H < G$.
THEN H IS QUASI-CONVEX IFF
H FIN GEN AND UNDISTORTED.

EXERCISE [SHORT 2000] SUPPOSE $H, K < G$ ARE QUASI CONVEX.
THEN SO IS $H \cap K$.

② CENTRALISERS

DEF: SUPPOSE $g \in G$. ~~DEFINE~~ THE CENTRALISER of g IS:

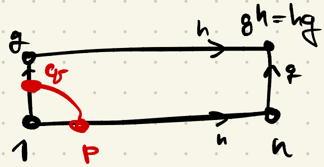
$$C_G(g) = \{ f \in G \mid fg = gf \}$$

EXERCISE: $C_G(g)$ IS A SUBGROUP of G. AND $g \in C_G(g)$ IS CENTRAL

EXERCISE: IF G ABELIAN, $C_G(g) = G$. IF $g = 1_G$, $C_G(g) = G$.
IF $g \in F(S)$ IS NOT A POWER THEN $C_{F(S)}(g) = \langle g \rangle$.

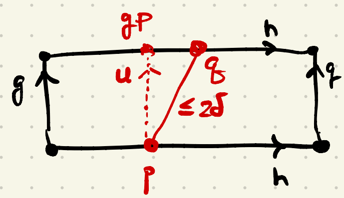
PROP: SUPPOSE (G,S) IS δ-HYP. THEN $C_G(g)$ IS QUASI-CONVEX.

PROOF: FIX $h \in C(g)$ AND CONSIDER QUAD



PICK ANY $p \in [1,h]$. SUPPOSE q LIES IN A g-SIDE of Q, AND $d_S(p,q) \leq 2\delta$.
THEN p IS WITHIN $2\delta + |g|_S$ of EITHER 1_G OR h. ✓

SUPPOSE q LIES IN $[g, gh]$.



EXERCISE: $d(p, gp) \leq 2\delta + |g|_S$.

SET $u = p^{-1} g p$. SO $|u| \leq 2\delta + |g|_S$.

SO THERE IS SOME $r \in G$ WITH $u = rgr^{-1}$ AND

$$|r|_S \leq |g|_S + |g|_S + 2\delta + K \text{ [COROLLARY]}.$$

SO: $p^{-1}gp = rgr^{-1}$ SO $g(pr) = (pr)g$ SO $pr \in C(g)$.
ALSO $d_S(p, pr) = |r|_S \leq 2|g|_S + 2\delta + K$. SO $C(g)$ IS $2|g|_S + 2\delta + K$ - QUASI-CONVEX. $\square$

OUR NEXT GOAL IS TO PROVE THE FOLLOWING:

THEOREM: SUPPOSE (G, S) IS δ-HYPERBOLIC.
THEN $\langle g \rangle < G$ IS QUASI-CONVEX (SO UNDISTORTED).

THIS IS SUPRISINGLY SUBTLE.

(2) QUASI-ISOMETRIC EMBEDDING AND QUASI-ISOMETRY

DEF: SUPPOSE $(X, d_X), (Y, d_Y)$ ARE METRIC SPACES. SUPPOSE $f: X \to Y$ A FUNCTION. WE SAY f IS AN (λ, c)-QUASI ISOMETRIC EMBEDING IF FOR ALL $x, x' \in X$ WE HAVE

$$\left.\begin{aligned} d_Y(y, y') &\leq \lambda\, d_X(x, x') + c \\ d_X(x, x') &\leq \lambda\, d_Y(y, y') + c \end{aligned}\right\} \quad \text{HERE } y = f(x),\ y' = f(x'),\ \lambda \geq 1,\ c \geq 0.$$

WE SAY f IS A (λ, c, D)-QUASI ISOMETRY IF, ADDITIONALLY, $N_Y(f(X), D) = Y$. [SAY $f(X)$ IS D-DENSE IN Y].

EXAMPLES:

(1) $Id_X : X \to X$ IS $\left\{\begin{array}{l} \text{A } (1,0) \text{ QI-EMBEDDING AND} \\ \text{A } (1,0,0) \text{ QUASI-ISOMETRY.} \end{array}\right.$

(2) $f: \mathbb{R} \to \mathbb{R}^2$ IS AN ISOMETRIC EMBEDDING, SO IS QI EMB. BUT IT IS NOT A QUASI-ISOMETRY.

(3) $\left.\begin{array}{l} f: \mathbb{Z} \to \mathbb{Z}^2 \\ n \longmapsto (2n, 0) \end{array}\right\}$ IS NOT ISOM. EMB. BUT IT IS A $(2,0)$-QI-EMB.

④ DEFINE $L : \mathbb{R} \longrightarrow \mathbb{Z}$ BY $L(x) = \max\{ k \in \mathbb{Z} \mid k \leq x \}$.

THEN L IS A $(1,1,0)$-QUASI ISOMETRY.