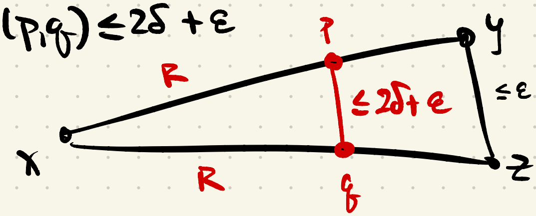
2025-02-27 LECTURE 24 SAUL SCHLEIMER MA4H4

① SHARP TRIANGLES

LEMMA: SUPPOSE (X, d_X) IS δ-HYPERBOLIC. SUPPOSE $x, y, z \in X$ WITH $d_X(y,z) \le \varepsilon$. SUPPOSE $0 \le R \le d_X(x,y), d_X(x,z)$. LET $p \in [x,y]$, $q \in [x,z]$ WITH $d_X(x,p) = d_X(x,q) = R$. THEN $d_X(p,q) \le 2\delta + \varepsilon$

PROOF: EXERCISE □

PICTURE

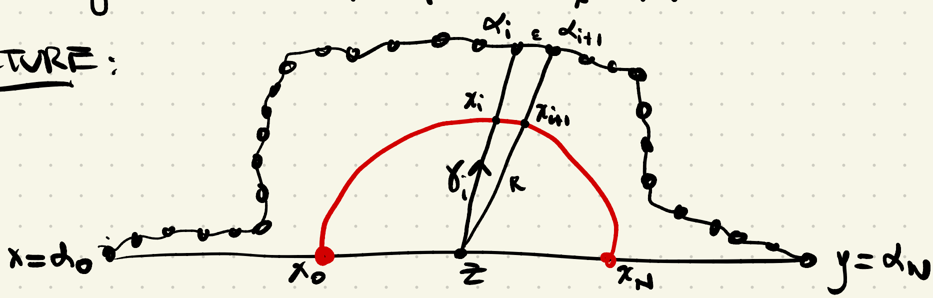


② DIVERGENCE

PROPOSITION: SUPPOSE (X, d_X) IS δ-HYP, WITH $\delta \ge 1$. SUPPOSE $[x,y]$ IS A GEODESIC, SUPPOSE $(\alpha_i)_{i=0}^{N}$ IS A SEQ of POINTS WITH $\alpha_0 = x$, $\alpha_N = y$, $d_X(\alpha_i, \alpha_{i+1}) \le \varepsilon$. FIX $z \in [x,y]$. SUPPOSE $d_X(z, \alpha_i) \ge R$ FOR ALL i. THEN $\boxed{\exp\left(\frac{R}{\delta} - 3 - \frac{\varepsilon}{\delta}\right) \le N}$

PROOF: WE FIX GEODESICS $\gamma_i = [z, \alpha_i]$, WITH $\gamma_0 = [z,x]$ $\gamma_N = [z,y]$ DEFINE $x_i \in \gamma_i$ BY $d_X(z, x_i) = R$.

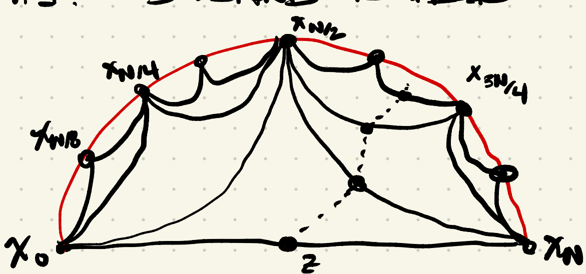
PICTURE:



THUS, $d_X(x_i, x_{i+1}) \le 2\delta + \varepsilon$ [LEMMA]. WE DEFINE GEODESIC TRIANGLES AS FOLLOWS:

CONNECT x_0, x_N BY GEODESIC.

IF x_i, x_j CONNECTED, AND IF $|j - i| \ge 2$ THEN



SET $k = \lfloor \frac{i+j}{2} \rfloor$ AND CONNECT x_i TO x_k AND x_k TO x_j

SO THERE IS A SEQ of $\lceil \log_2(N) \rceil + 1$ GEOD ARCS of LENGTH $\leq \delta$ FROM z TO SOME $[x_i, x_{i+1}]$. SO

$$R \leq d_X(z, x_i) \leq \delta \log_2 N + \delta + 2\delta + \varepsilon$$

$$\leq \delta \log_2 N + 3\delta + \varepsilon$$

SO $$\mathrm{EXP}\left(\frac{R}{\delta} - 3 - \frac{\varepsilon}{\delta}\right) \leq N$$ $\square$

COROLLARY: SUPPOSE (G,S) IS δ-HYPERBOLIC AND NOT VIRTUALLY CYCLIC. THEN G HAS EXP. GROWTH.

PROOF: IF $e(G) = 1$ THEN USE PROPOSITION. IF $e(G) = \infty$ THEN [EXERCISE] $\square$

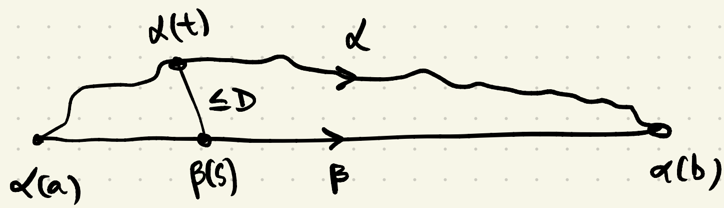
③ QUASI-GEODESICS

DEF: A QUASI-GEODESIC IS A QUASI-ISOMETRIC EMBEDDING OF AN INTERVAL, RAY, OR LINE (of $\mathbb{R}$ OR of $\mathbb{Z}$).

THEOREM [STABILITY] FOR ANY δ, λ, c THERE IS $D = D(\delta, \lambda, c)$ AS FOLLOWS. SUPPOSE X IS δ-HYP. SUPPOSE $\alpha : [a,b] \to X$ IS A (λ, c)-QUASI-GEODESIC. SUPPOSE $\beta = [\alpha(a), \alpha(b)]$ IS A GEODESIC. THEN: $\alpha \subset N_X(\beta, D)$ AND $\beta \subset N_X(\alpha, D)$.

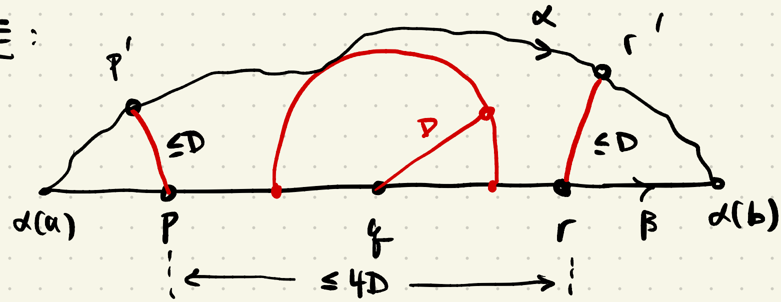
[HERE WE USE α AS SHORTHAND FOR $\alpha([a,b])$].

PICTURE:



PROOF: SUPPOSE $q \in \beta$ MAXIMISES $\{ d_X(q', \alpha) \mid q' \in \beta \}$.
SET $D = d_X(q, \alpha)$. THUS $\beta \subset N_X(\alpha, D)$. PICK $p, r \in \beta$
SO THAT: $d_X(p,q) = d_X(q,r) = 2D$ AND $p < q < r$ ALONG β.
[IF p OR r NOT WELL DEF, REPLACE BY $\alpha(a), \alpha(b)$ AS NEEDED]
LET $p', r' \in \alpha$ BE CLOSEST TO p, r.

PICTURE:



SO $d_X(p', r') \leq 6D$.
FIX a', b' SO $\alpha(a') = p'$, $\alpha(b') = r'$. SO $|b' - a'| \leq \lambda 6D + C$
CONCATENATE TO GET $\alpha' = [p, p'] * \alpha|[a', b'] * [r', r]$
TAKE THE CORNERS p, p', r', r AND INTEGER POINTS TO
GET SEQUENCE $\{\alpha'_i\}_{i=0}^{N}$ WITH $N \leq \lambda 6D + C + 2D + 4$, WITH
$\alpha'_0 = p$, $\alpha'_N = q$, AND WITH $d_X(\alpha'_i, \alpha'_{i+1}) \leq \max\{1, \lambda + C\}$
SET $\varepsilon = \max\{1, \lambda + C\}$. BY PROPOSITION

$$\exp\left(\frac{D}{\delta} - 3 - \frac{\varepsilon}{\delta}\right) \leq (\lambda 6 + 2)D + C + 4 .$$

LHS GROWS EXP'LY, RHS LINEARLY. SO D BOUNDED
IN TERMS OF λ, C, δ.

EXERCISE: FIND D' SO $\alpha \subset N_X(\beta, D')$. $\square$