2025-03-04 LECTURE 25 SAUL SCHLEIMER MA4H4

① **EXERCISES**: SUPPOSE (G,S) IS FIN. GEN.
SUPPOSE $H < G$ HAS INDEX $d < \infty$.

THEN
1) $G = N_S(H, d-1)$ (EX 5.5)
2) H IS FIN GEN (EX 4.5(1))
3) $e(G) = e(H)$ (EX 5.1).

② QUASI-GEODESIC STABILITY.

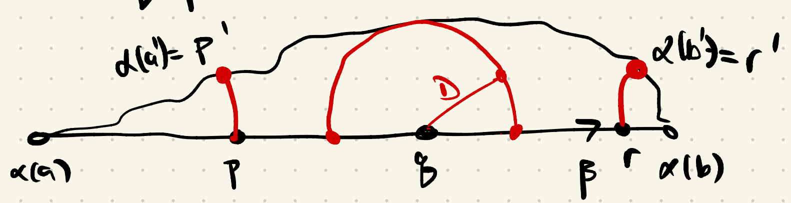
WE WERE PROVING:

THEOREM: FIX $\delta \geq 0$, $\lambda \geq 1$, $c \geq 0$. THEN THERE IS A CONSTANT $D = D(\delta, \lambda, c)$ AS FOLLOWS. SUPPOSE (X, d_X) IS δ-HYP AND $\alpha : [a,b] \longrightarrow X$ IS A (λ, c)-QUASI-GEODESIC. SUPPOSE $\beta = [\alpha(a), \alpha(b)]$ IS A GEODESIC. THEN

$$\alpha \subset N_X(\beta, D) \quad \text{AND} \quad \beta \subset N_X(\alpha, D)$$

(WHERE $\alpha = \alpha([a,b])$ IS THE IMAGE).

PROOF: $q \in \beta$ FURTHEST FROM α, p, r ON β, p', r' ON α AS SHOWN.



SO $d_X(p', r') \leq 6D$. SO $|b' - a'| \leq \lambda 6D + c$, BECAUSE α IS A QUASI-GEOD. WE TAKE $(\alpha_i')_{i=0}^{N}$ TO BE THE CORNERS AND INTEGER POINTS of

$$[p, p'] \cup \alpha|[a', b'] \cup [r', r]$$

SO: $\alpha_0 = p$, $\alpha_N = r$, $N \leq 4 + 2D + 2 + |b' - a'| + 1$

$$\leq 2D + \lambda 6D + c + 7.$$

AND $d_X(\alpha'_i, \alpha'_{i+1}) \le \lambda + C$. SET $\varepsilon = \lambda + C$.

APPLY PROPOSITION TO FIND

$$\exp\left(\frac{D}{\delta} - 3 - \frac{\varepsilon}{\delta}\right) \le N \le (2+\lambda 6)D + C + 7.$$

SO FIND D BOUNDED BY SOME POLY IN δ, λ, C.

THUS $\beta \subset N_X(\alpha, D)$.

<u>EXERCISE:</u> PROVE (FOR SOME $D' = D'(\delta, \lambda, C)$) THAT $\alpha \subset N_X(\beta, D')$.

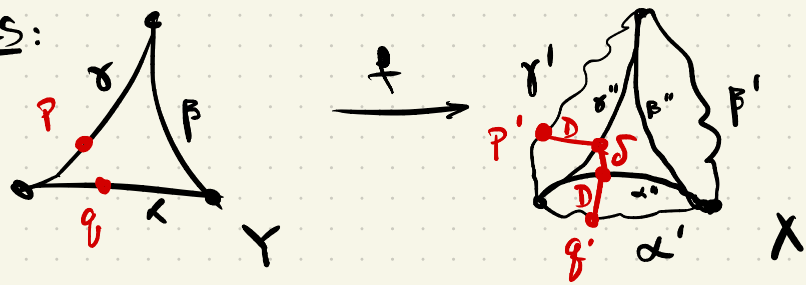
$\square$

③ <u>HYPERBOLICITY IS QI-INVAR.</u>

<u>THEOREM</u>: SUPPOSE (X, d_X), (Y, d_Y) GEOD. METRIC SPACES, X δ-HYP, AND $f: Y \to X$ A (λ, C) QI EMBEDDING. THEN Y IS δ'-HYP FOR SOME δ'.

<u>PROOF</u>: SUPPOSE $T = (\alpha, \beta, \gamma)$ IS A GEOD TRIANGLE IN Y. SO $\alpha' = f \circ \alpha$, $\beta' = f \circ \beta$, $\gamma' = f \circ \gamma$ IS A QUASI-GEOD TRIANGLE IN X. LET $\alpha'', \beta'', \gamma''$ BE GEOD IN X WITH SAME ENDPTS. SO $T'' = (\alpha'', \beta'', \gamma'')$ IS δ-SLIM. SO $T' = (\alpha', \beta', \gamma')$ IS $\delta + 2D$-SLIM. SO $T = (\alpha, \beta, \gamma)$ IS $\lambda(\delta + 2D) + C$-SLIM $\square$

<u>PICTURES</u>:



<u>COROLLARY</u> HYPERBOLICITY IS A QI INVARIANT. $\square$

③ <u>MANY EXERCISES</u> SUPPOSE X, Y, Z METRIC SPACES.

① IF $f: X \to Y$, $g: Y \to Z$ ARE QI-EMBEDDINGS THEN SO IS

$$g \circ f : X \longrightarrow Z.$$

② QUASI-ISOMETRY IS AN EQUIV. RELATION ON METRIC SPACES

(3) $\mathbb{Z}^n$ QI TO $\mathbb{R}^n$.

(4) $(\mathbb{R}^n, L^p)$ QI TO $(\mathbb{R}^n, L^q)$ FOR $1 \leq p, q \leq \infty$.

(5) IF $X \underset{QI}{\cong} X'$, $Y \underset{QI}{\cong} Y'$ THEN $X \times Y \underset{QI}{\cong} X' \times Y'$.

SUPPOSE G FIN GEN BY S, T.

(6) $\Gamma(G,S)$ IS QI TO $\Gamma(G,T)$.

SUPPOSE $(G,S), (H,T)$ FIN GEN GROUPS.

SUPPOSE $\Gamma(G,S)$ IS QI TO $\Gamma(H,T)$.

(7) $e(G) = e(H)$.

(8) $\gamma_G \sim \gamma_H$ [EQUIV GROWTH RATES].

(9) G VIRT NILPOTENT IFF H IS.

(10) G VIRT ABELIAN IFF H IS. [HARDER]

(11) G FIN PRES IFF H IS. [HARDER]

NOTE THAT (9) DOES NOT EXTEND TO SOLVABLE GROUPS.

<u>THEOREM</u> [DIOUBINA, 2000]

THERE ARE FIN. GEN. GROUPS G, H SO THAT $G \underset{QI}{\cong} H$, G IS SOLVABLE, AND H IS NOT VIRT. SOLVABLE.