① THE FUNDAMENTAL OBSERVATION of GEOMETRIC GROUP THY

DEF: SUPPOSE X IS A METRIC SPACE. SUPPOSE G ACTS ON X BY ISOMETRIES. THE ACTION IS COCOMPACT IF THE QUOTIENT SPACE $G\backslash X$ IS COMPACT. THE ACTION IS PROPERLY DISCONTINUOUS IF FOR ALL $K \subset X$ COMPACT WE HAVE $\{g \in G \mid K \cap gK \neq \emptyset\}$ IS FINITE. THE ACTION IS GEOMETRIC IF X IS PROPER, GEODESIC AND G ACTS COCOMPACTLY AND PROPERLY DISCONTINUOUSLY.

EX: $\mathbb{Z}$ ACTING ON $\mathbb{R}$ IS GEOMETRIC

$\mathbb{R}$ ACTING ON $\mathbb{R}$ IS COCOMPACT BUT NOT PROP. DISCONTIN

G ACTING ON $\Gamma(G,S)$ IS GEOMETRIC IFF S IS FINITE.

DEF: SUPPOSE $G : X \circlearrowleft$ IS A GEOMETRIC ACTION. DEFINE $X' = G\backslash X$ AND

$$d_{X'}([x],[y]) = \inf\{ d_X(g\cdot x, h\cdot y) \mid g,h \in G\}$$

LEMMA: (1) THE INF IS REALISED.

(2) $(X', d_{X'})$ IS A GEODESIC METRIC SPACE. THE METRIC TOPOLOGY EQUALS THE QUOTIENT TOPOLOGY.

PROOF: EXERCISES. □

LEMMA [MILNOR-ŠVARC] SUPPOSE $G : X \circlearrowleft$ IS A GEOMETRIC ACTION. THEN

(1) G IS FIN GEN AND

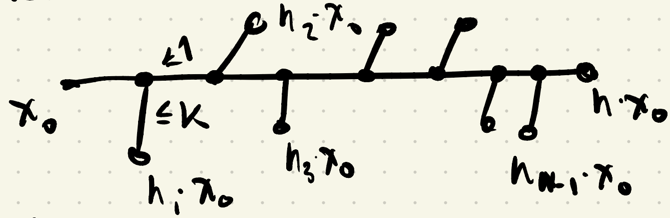
(2) FOR ANY FIN GEN SET $S \subset G$, FOR ANY $x_0 \in X$, THE MAP $g \longmapsto g\cdot x_0$ IS A QUASI-ISOMETRY FROM (G, d_S) TO (X, d_X)

PROOF: SET $X' = G\backslash X$. LET $[x] \in X'$ BE THE CLASS of x.
SINCE X' COMPACT IT HAS BOUNDED DIAMETER, SAY K.
FIX $x_0 \in X$.

CLAIM: $G \cdot x_0$ IS K-DENSE IN X.

PROOF: FIX ANY $y \in X$. SO $d_{X'}([x_0],[y]) \leq K$. SO THERE ARE $g, h \in G$ WITH $d_X(g \cdot x_0, h \cdot y) \leq K$.
SO $d_X((h^{-1}g) \cdot x_0, y) \leq K$. $\square$

SUPPOSE $h \in G$. FIX GEODESIC $\alpha = [x_0, h \cdot x_0]$.
LET $(\alpha_i)_{i=0}^{N}$ BE THE END- AND INTEGER-POINTS of α.



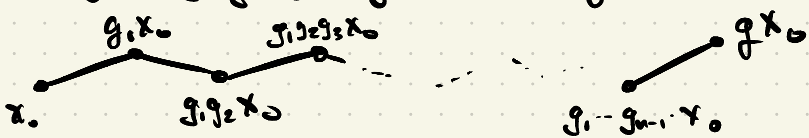
SO HAVE $(h_i) \subset G$ WITH $h_0 = 1_G$, $h_N = h$ AND
$d_X(\alpha_i, h_i x_0) \leq K$. SO $d_X(h_i x_0, h_{i+1} x_0) \leq 2K+1$.
SO FOR ALL i HAVE $d_X(x_0, (h_i^{-1} h_{i+1}) \cdot x_0) \leq 2K+1$.

$B = B_X(x_0, 2K+1)$ IS A CLOSED BALL SO COMPACT.
SO $S = \{ g \in G \mid B \cap gB \neq \emptyset \}$ IS FINITE. THIS IS THE DESIRED FIN GEN SET FOR G. (1)

BY LEMMA [2025-01-22] ALL WORD METRICS ON G ARE COMPARABLE. SO SUFFICES TO PROVE $(G, d_S) \to (X, d_X)$ IS A QUASI-ISOMETRY.

SUPPOSE $g \in F(S)$ IS A GEOD WORD FOR (G,S).
SUPPOSE $g = g_1 g_2 g_3 \cdots g_n$ FOR $g_i \in S$, $n = |g|_S$

SO



IS A PATH IN X. SO $d_X(x_0, g\cdot x_0) \leq (2K+1)\cdot |g|_S$.

<u>EXERCISE</u>: $|g|_S \leq d_X(x_0, g\cdot x_0) + 1$

[HINT: CONSIDER $\alpha = [x_0, g\cdot x_0]$] $\square$

---

(2) CORROLARIES/EXERCISES

1) WITH F_n THE FREE GROUP of RANK n: F_m IS QI TO F_n FOR ALL $m,n \geq 2$. [OR COULD USE FINITE INDEX SUBGPS...]

2) SUPPOSE $g \geq 2$. THEN $\pi_1(S_g)$ QI TO $\mathbb{H}^2$.
   MORE GENERALLY, IF M IS A CLOSED, CONN, RIEMANNIAN n-MANIFOLD, THEN $\pi_1(M)$ IS QI TO $\tilde{M}$ [UNIV. COVER].

3) SUPPOSE $H < G$ FIN INDEX. SUPPOSE S, T ARE FIN. GEN SETS FOR G, H. THEN $i_H : (H, d_T) \to (G, d_S)$ IS QI. (*) ALLOW $G = H$.

4) SUPPOSE $H < G$ IS FIN GEN BY T, S RESPECTIVELY. THEN $i : (H, d_T) \longrightarrow (G, d_S)$ IS QI-EMB IFF H IS UNDISTORTED.

5) $\pi_1(S_g)$ IS QI TO F_n IFF $g = n = 0$.

---

(3) CYCLIC SUBGROUPS ARE UNDISTORTED

<u>THEOREM</u>: SUPPOSE (G,S) IS δ-HYP. SUPPOSE $g \in G$ IS NOT TORSION. THEN

$$\mathbb{Z} \longrightarrow G \quad \text{IS A QI EMBEDDING.}$$
$$n \longmapsto g^n \quad [\langle g \rangle \text{ IS UNDISTORTED}].$$

<u>PROOF</u>: RECALL $C_G(g)$ IS QUASI-CONVEX, SO IS FIN GEN, SAY BY $T \subset C_G(g)$.

<u>EXERCISE</u> $Z(C_G(g)) = \bigcap_{t \in T} C_G(t)$ IS THE CENTRE of $C_G(g)$, AND IS QUASI-CONVEX. SO $Z = Z(C_G(g))$ IS ABELIAN, IS FIN GEN, IS UNDISTORTED.

SO $Z \cong \mathrm{TOR} \oplus \mathbb{Z}^k$. SO $\mathbb{Z}^k < Z$ IS FIN INDEX.

SO $\mathbb{Z}^k < Z < G$ IS UNDISTORTED. SO $k=1$
[AS $\mathbb{Z} \cong \langle g \rangle < Z$]. $\square$