SUPP 9 MB0.08
SUPP 10 C1.14

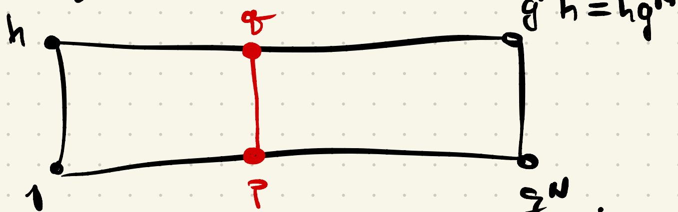
① **THEOREM**: SUPPOSE (G,S) IS δ-HYP. SUPPOSE $g \in G$ IS NOT TORSION. **THEN** $C_G(g)$ IS VIRT. CYCLIC.

PROOF: PICK $h \in C_G(g)$. PICK N SO THAT $|g^N|_S > 2|h|_S + 2\delta$

NOTE THIS IS POSS. BECAUSE (g^n) IS A QUASI-GEOD. [THEOREM]

SO THERE IS SOME POINT $p \in [1, g^N]$ WHICH IS 2δ FAR FROM $[1,h]$ AND $[g^N, g^N h]$. SO p IS 2δ-CLOSE TO SOME POINT $q \in [h, hg^N]$

PICTURE



BY STABILITY, THERE ARE i,j SO $d_S(p, g^i) \le D$

$d_S(q, hg^j) \le D$.

SO $d_S(g^i, g^j h) \le 2D + 2\delta$. SO $d_S(1_G, g^{j-i} h) \le 2D + 2\delta$.

SO EVERY COSET $\langle g \rangle \cdot h \in \langle g \rangle \backslash C_G(g)$ HAS A REP IN $B_S(1_G, 2D+2\delta)$ WHICH IS FINITE. □

COROLLARY: SUPPOSE (G,S) IS δ-HYP. THEN $\mathbb{Z}^2$ DOES NOT EMBED IN G.

PROOF: SUPPOSE $g, h \in G$ AND $gh = hg$. SO $h \in C_G(g)$.

SO $[h]$ IS FINITE ORDER IN $\langle g \rangle \backslash \langle h, g \rangle \subset \langle g \rangle \backslash C_G(g)$. □

COROLLARY: SUPPOSE (G,S) IS δ-HYP. THEN $BS(p, \pm p)$ $(p \neq 0)$ DOES NOT EMBED IN G. □

2 AVERAGE TRANSLATION DISTANCE

DEF: SUPPOSE (G,S) IS FIN. GEN. DEFINE $\tau_S : G \longrightarrow \mathbb{R}_{\geq 0}$ BY

$$\tau_S(g) = \lim_{n\to\infty} \frac{1}{n} d_S(1_G, g^n).$$

WE CALL THIS THE <u>AVERAGE TRANSLATION DIST</u> of g IN Γ_S.

EXERCISE:

(1) $\tau_S(g)$ IS WELL-DEF.

(2) $\tau_S(g^m) = |m| \cdot \tau_S(g)$ FOR $m \in \mathbb{Z}$.

(3) IF g CONJ TO h THEN $\tau_S(g) = \tau_S(h)$

(4) SUPPOSE (G,S) IS δ-HYP.
 <u>THEN</u> $\tau_S(g) = 0$ IFF g IS TORSION

COROLLARY: SUPPOSE (G,S) δ-HYP. THEN $BS(p,q)$ DOES NOT EMBED IN G.

PROOF. SUPPOSE $BS(p,q) = \langle a, b \mid ab^p = b^q a \rangle$ EMB IN G. SO b NON TORSION, SO $\tau_S(b) > 0$. SINCE $ab^pa^{-1} = b^q$ WE HAVE

$$\begin{aligned} |p| \cdot \tau_S(b) &= \tau_S(b^p) \\ &= \tau_S(b^q) \\ &= |q|\, \tau_S(b). \end{aligned}$$

SO $|p| = |q|$ SO $q = \pm p$. APPLY COROLLARY. $\square$.

OPEN: SUPPOSE (G,S) IS of 'FINITE TYPE' ($G = \pi_1(C, c_0)$ WHERE C FINITE CW AND $\tilde{C}$ CONTRACTIBLE). THEN G IS δ-HYP IFF G HAS NO $BS(p,q)$ SUBGROUPS.

[<u>THAT IS</u>: FOR 'NICE' GROUPS, FLATS AND DISTORTED CYCLIC SUBGROUPS ARE THE <u>ONLY</u> OBSTRUCTIONS TO HYPERBOLICITY.]

OOPS! NOT OPEN AS of 2021. SEE 2025-03-11.