<u>NOT OPEN</u>: SUPPOSE (G,S) IS OF "FINITE TYPE". IF G HAS NO $BS(p,q)$ SUBGROUPS THEN G IS δ-HYP

<u>OOPS!</u> SEE ITALIANO, MARTELLI, MIGLIORINI (2021) "HYP 5-MFDS THAT FIBER OVER S^1" FOR COUNTEREXAMPLES!

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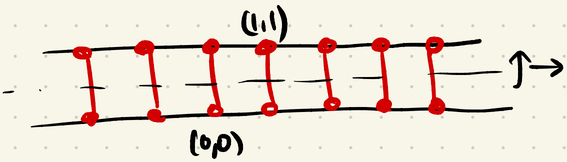
① <u>AXES</u>: SUPPOSE (G,S) IS δ-HYP. SUPPOSE $g \in G$. A GEODESIC $A \subset \Gamma(G,S)$ IS AN <u>AXIS</u> FOR g IF $g \cdot A = A$ AND g ACTS ON A BY NON-TRIV TRANSLATION.

<u>EXAMPLE</u>: $\{a^n \mid n \in \mathbb{Z}\} \subset F(a,b)$ GIVES AN AXIS FOR a

<u>LEMMA</u>: ALL AXES FOR $g \in G$ 2δ FELLOW TRAVEL [IF THEY EXIST AT ALL]

<u>EXAMPLE</u>: IN $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ THE ELEMENT $(1,1)$ HAS NO AXIS.



$(1,1)$ ACTS VIA GLIDE REFLECTION.

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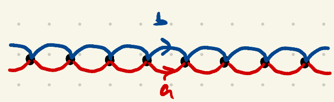
② <u>RATIONAL TRANS. LENGTH</u> [DELZANT]

FIX (G,S) A FIN. GEN GROUP.

<u>DEF</u>: AN EDGE PATH $\gamma : \mathbb{R} \to \Gamma_S$ IS A <u>SHORT-LEX GEODESIC</u> IF FOR ALL $m, n \in \mathbb{Z}$, THE LABEL OF $\gamma|[m,n]$ IS A SHORT-LEX WORD.

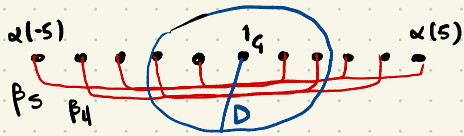
<u>EXAMPLE</u>: $\mathbb{Z} \cong \langle a, b \mid a = b \rangle$



WITH $a < b$.

<u>LEMMA</u> [EXISTENCE] SUPPOSE (G,S) IS δ-HYP. SUPPOSE $\alpha : \mathbb{Z} \to G$ IS A (λ, c) QUASI GEODESIC. LET $D = D(\delta, \lambda, c)$ BE THE STABILITY CONSTANT. THEN THERE EXISTS A BIINFINITE SHORT-LEX GEODESIC CONTAINED IN THE D-NEIGHBOURHOOD OF α.

PROOF: LET β_n BE THE SHORT LEX GEODESIC FROM $\alpha(-n)$ TO $\alpha(n)$. SO $|\beta_n| \geq \frac{1}{\lambda}(2n-c)$. NOTE EVERY β_n MEETS $B_S(1_G, D)$. ALSO, FOR $n \geq \lambda D + C$, THE ENDPOINTS OF β_n ARE OUTSIDE OF $B_S(1,D)$. SO, PICK AN INF SUBSEQ $B_1 \subset B_0$ SO THAT $\beta_i \cap B_S(1,D) = \beta_j \cap B_S(1,D)$ FOR ALL $\beta_i, \beta_j \in B_1$. THAT B_1 EXISTS FOLLOWS FROM THE INF TO FIN PIGEONHOLE PRINCIPLE. PICTURE



DEFINE $\gamma_1 = \beta_n \cap B_S(1,D)$ FOR ANY $\beta_n \in B_1$. AT STAGE k WE HAVE B_k, γ_k WITH $B_k \subset B_0$ INF, WITH $\gamma_k = \beta_i \cap B_S(1, D+k)$ FOR ALL $\beta_i \in B_k$, AND WITH $\gamma_{k-1} \subset \gamma_k$. BY INF-TO-FINITE PIGEONHOLE PRINCIPLE WE CAN FORM B_{k+1} AND γ_{k+1} SIMILARLY.

SET $\gamma = \bigcup_k \gamma_k$. THIS IS AN ASCENDING UNION OF SHORT LEX GEODS, D-CLOSE TO α, SO IS ITSELF A (BIINFINITE!) SHORT-LEX GEOD. D-CLOSE TO α. $\square$

WE LEAVE THE PROOFS OF THE FOLLOWING AS EXERCISES.

PROPOSITION: SUPPOSE (G,S) IS δ-HYP. SUPPOSE $g \in G$ IS NOT TORSION. LET (λ, c) BE THE QUASI GEOD CONST FOR $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ THEN THERE ARE AT MOST $V_{2\delta} = \mathrm{CARD}(B_S(1_G, 2\delta))$ MANY BIINF SHORT-LEX GEOD. THAT LIE IN THE D-NEIGHBOURHOOD OF $\langle g \rangle$. ANY TWO SUCH MEET IFF THEY ARE EQUAL.

COROLLARY [DELZANT] SUPPOSE (G,S) IS δ-HYPERBOLIC. SUPPOSE $g \in G$ IS NOT TORSION. THEN THERE IS $N \leq V_{2\delta}$ SO THAT g^N HAS A SHORTLEX AXIS.

COROLLARY THUS $\tau(g^N) \in \mathbb{Z}$ AND $\tau(g) \in \frac{1}{N}\mathbb{Z}$.

RESTATED: A FIX HYPERBOLIC GROUP HAS UNIFORMLY RATIONAL AVERAGE TRANSLATION LENGTHS.

[CONNER 1997] GIVES FIN GEN GROUPS (of FORM $\mathbb{Z}^k \rtimes \mathbb{Z}$) HAVING SOME ELTS WITH IRRAT. AVE TRANS LENGTH.

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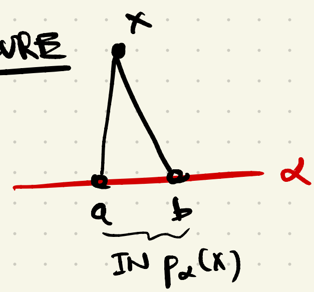
② CLOSEST POINTS PROJECTION

SUPPOSE (X, d_X) IS PROPER, δ-HYP. SUPPOSE $\alpha \subset X$ IS A GEODESIC. WE DEFINE $p_\alpha : X \longrightarrow \alpha$ BY TAKING $p_\alpha(x) \in \alpha$ TO MINIMISE $d_X(x, p_\alpha(x))$. NOTE THAT p_α MAY NOT BE WELL DEFINED. SO WE INSTEAD TAKE $p_\alpha(x)$ TO BE A SUBSET of α.

PICTURE



LEMMA: $\mathrm{DIAM}_X(p_\alpha(x)) \leq 4\delta$.

SO $p_\alpha : X \longrightarrow \alpha$ IS A QUASI-FUNCTION [ONLY HAS IMAGES UP TO UNIF. ADDITIVE ERROR].
