2025-03-12 LECTURE 29 MA4H4

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$\boxed{\text{SUPP 10} \quad \text{C1.14}}$

① ELEMENTARY HULLS: SUPPOSE G A GROUP, $g \in G$.

DEFINE: $E(g) = \{ h \in G \mid h g^n h^{-1} = g^{\pm n} \text{ FOR SOME } n \neq 0 \}$.

$E^+(g) = \{ h \in G \mid h g^n h^{-1} = g^n \text{ FOR SOME } n \neq 0 \}$.

WE CALL $E(g)$ THE ALMOST-CENTRALISER of $g \in G$.

EXERCISES [HULL, 2015]

(1) $E(g) < G$ IS A SUBGROUP

(2) $E^+(g) < E(g)$ HAS INDEX TWO (OR ONE)

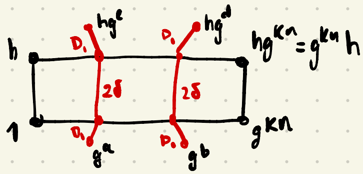
(3) $E(g) = E(g^k)$ FOR ANY $k \neq 0$.

(4) $\langle g \rangle < C_G(g) < E^+(g) < E(g)$

LEMMA: $\langle g \rangle$ IS FIN INDEX IN $E(g)$. ALSO, $E(g)$ IS MAXIMAL VIRT CYC. SUBGP of G CONTAINING $\langle g \rangle$ [AND IS UNIQUE SUCH].

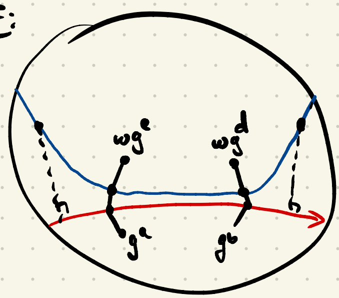
FURTHERMORE: THERE IS $N = N(g)$ SO THAT $E^+(g) = C_G(g^N)$

PROOF:



PROP: SUPPOSE (G,S) δ-HYP. SUPPOSE g NON TORSION. SUPPOSE $w \notin S_G(g)$. SUPPOSE α GEOD BOUNDED DIST TO/FROM $\langle g \rangle$. SET $\beta = w \cdot \alpha$. THEN $p_\alpha(\beta)$ AND $p_\beta(\alpha)$ BOUNDED DIAMETER.

PROOF:



SUPPOSE $p_\alpha(\beta)$ HAS LARGE DIAMETER.

SO $d_S(g^a, wg^c), d_S(g^b, wg^d)$ SMALL.

SO REPEAT [AGAIN, PIGEONHOLE]

SO $g^{-a} w g^c = g^{-b} w g^d$

SO $g^{b-a} w = w g^{d-c}$.

SO $b - a = \pm(d-c)$ AND $w \in S_G(g)$ ∎

② **PINGPONG LEMMA:** SUPPOSE G IS A GROUP, ACTING ON A SET X. SUPPOSE $a, b \in G$ ARE ELEMENTS. SUPPOSE $X_a, X_b, X_{a^{-1}}, X_{b^{-1}}$ ARE DISJOINT SUBSETS of X.

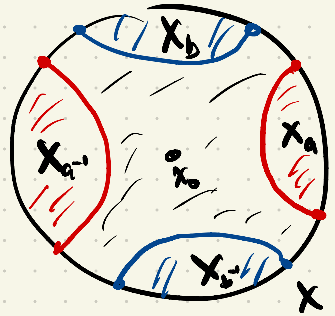
SUPPOSE $x_0 \in X - (X_a \cup X_b \cup X_{a^{-1}} \cup X_{b^{-1}})$.

SUPPOSE $a \cdot (X - X_{a^{-1}}) \subset X_a$.

$b \cdot (X - X_{b^{-1}}) \subset X_b$

$a^{-1} (X - X_a) \subset X_{a^{-1}}$

$b^{-1} (X - X_b) \subset X_{b^{-1}}$



~~THEN~~ THEN THE SUBGROUP $\langle a, b \rangle < G$ IS FREE: $\langle a, b \rangle \cong F(a, b)$

PROOF: SET $S = \{a, b\}$. PICK $w \in F(S)$. SET $y = w \cdot x_0$. SO w FREELY REDUCED. SUPPOSE $w = w_n w_{n-1} \cdots w_2 w_1$

SET $x_k = (w_k w_{k-1} \cdots w_2 w_1) \cdot x_0$. IF $k = 0$. $x_k = x_0$.

INDUCTIVELY, $x_k \in X_{w_k}$. SO $x_{k+1} = w_{k+1} x_k \in X_{w_{k+1}}$.

SO $w \cdot x_0 = x_0$ IFF $w = \varepsilon_S$. SO $\langle a, b \rangle$ IS FREE. ∎.

③ **FREE SUBGROUPS of HYP GROUPS.**

THEOREM: SUPPOSE (G, S) δ-HYP. THEN EITHER G IS VIRT CYCLIC (OR FINITE) OR THERE ARE $a, b \in G$ SO $\langle a, b \rangle$ IS FREE, RANK TWO.

COMPARE:

TITS ALTERNATIVE: SUPPOSE $G < GL(n, k)$ IS FIN. GEN. THEN EITHER G IS VIRT SOLVABLE OR THERE ARE $a, b \in G$ SO $\langle a, b \rangle$ IS FREE, RANK TWO.