2025-03-13 LECTURE 30 MA4H4 SAUL SCHLEIMER

SUPP 10 C1.14

① FREE SUBGROUPS of HYP GROUPS.

THEOREM: SUPPOSE (G,S) δ-HYP. SUPPOSE $e(G) = 1$ OR ∞. THEN THERE ARE $a, b \in G$ SO $\langle a, b\rangle$ IS FREE, RANK TWO.

PROOF:

$e(G) \neq 0$ SO G INF. SO G HAS SOME NON TORS. ELT SAY g.

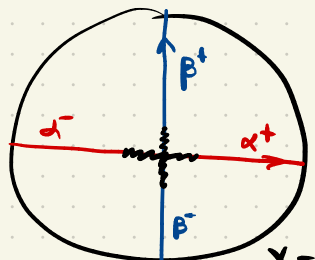
$e(G) \neq 2$ SO G NOT VIRT CYCLIC. SO PICK $w \in G - S_G(g)$.

SET $h = wgw^{-1}$. LET $\alpha = A_g$, $\beta = A_h$ AXES FOR g AND h. LET $\alpha^+, \alpha^- \subset \alpha$, $\beta^+, \beta^- \subset \beta$ BE SUBRAYS

SO $\alpha^+ \cap p_\alpha(\beta) = \alpha^- \cap p_\alpha(\beta) = \emptyset$.

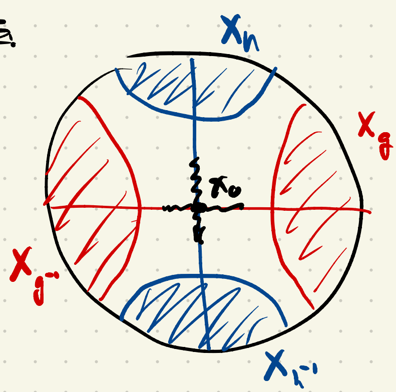
$\beta^+ \cap p_\beta(\alpha) = \beta^- \cap p_\beta(\alpha) = \emptyset$.

DEFINE
$$X_g = p_\alpha^{-1}(\alpha^+)$$
$$X_h = p_\beta^{-1}(\beta^+)$$
$$X_{g^{-1}} = p_\alpha^{-1}(\alpha^-)$$
$$X_{h^{-1}} = p_\beta^{-1}(\beta^-)$$



$X = \Gamma(G,S)$.

PICTURE



SHRINK $\alpha^+, \alpha^-, \beta^+, \beta^-$ TO ENSURE $X_g, X_h, X_{g^{-1}}, X_{h^{-1}}$ ALL DISJOINT. TAKE POWERS $a = g^N$, $b = h^N$ TO ENSURE HYPOTHESES of PING-PONG LEMMA. □.

---

② HAUSDORFF DISTANCE

SUPPOSE (X, d_X) IS A METRIC SPACE. SUPPOSE $A, B \subset X$.

DEFINE

$$d_X^{HAUS}(A,B) = \inf\{R \geq 0 \mid A \subset N_X(B,R) \text{ AND } B \subset N_X(A,R)\}.$$

WE SET $\inf(\emptyset) = +\infty$.

WE CALL $d_X^{HAUS}(A,B)$ THE <u>HAUSDORFF DISTANCE</u> BETWEEN A AND B.

<u>EXAMPLE</u>: IF $A = \{a\}$, $B = \{b\}$ ARE SINGLETONS, THEN $d_X^{HAUS}(A,B) = d_X(a,b)$.

③ <u>THE GROMOV BOUNDARY</u>: SUPPOSE (X, d_X) IS δ-HYP, PROPER

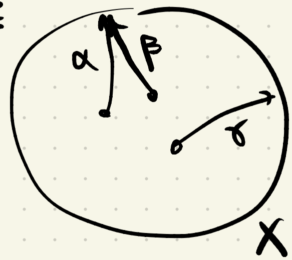
DEFINE $G(X) = \{\alpha : \mathbb{R}_{\geq 0} \to X \text{ GEOD. RAY}\}$

FOR $\alpha, \beta \in G(X)$ WRITE $\alpha \sim \beta$ IF $d_X^{HAUS}(\alpha, \beta) < \infty$.

EXERCISE: THIS IS AN EQU. RELATION.

DEFINE $\partial_\infty X = G(X)/_\sim$. THIS IS (AN) UNDERLYING SET FOR THE <u>GROMOV BOUNDARY</u>.

<u>PICTURE</u>



$\alpha \sim \beta$

$\alpha \not\sim \gamma$.

THERE IS A TOPOLOGY ON $\partial_\infty X$: IF $\alpha(0) = \beta(0)$ THEN α, β ARE $1/R$-CLOSE IF $d_X^{HAUS}(\alpha|_{[0,R]}, \beta|_{[0,R]})$

IS SMALL.

<u>EXAMPLES</u>: SUPPOSE (G,S) δ-HYP. ABUSE NOTATION AND USE $G = \Gamma$

(1) $\partial_\infty G = \emptyset$ IFF G FINITE

(2) $\partial_\infty \mathbb{Z} = \{-\infty, +\infty\}$ HAS TWO PTS.

(3) $\partial_\infty F(a,b) \cong \mathcal{C}$ IS A CANTOR SET

(4) $\partial_\infty \pi_1(\Sigma_2) \cong S^1$ IS A CIRCLE.

THEOREM: $\#(\text{PATH COMPONENTS } \partial_\infty G) = e(G)$.

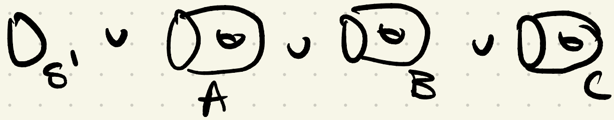
THEOREM: IF $f : \Gamma(G,S) \to \Gamma(H,T)$ IS QISOM THEN

$\partial_\infty f : \partial_\infty G \to \partial_\infty H$ IS A HOMEOMORPHISM.

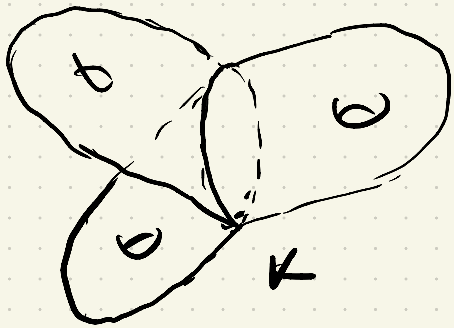
EXAMPLE

$G = \langle a,b,c,d,e,f \mid [a,d] = [b,e] = [c,f] \rangle = \pi_1(K)$

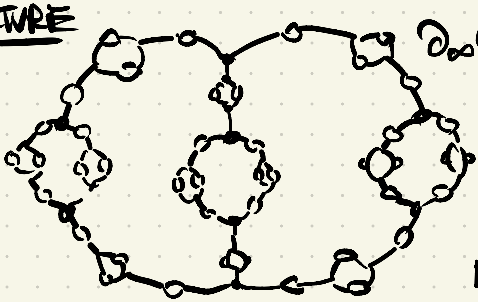
WITH $K =$



GLUE $\partial A, \partial B, \partial C$ TO S^1 VIA HOMEOS TO GET K: A "BOOK" of THREE "PAGES". **PICTURE**



PICTURE



$\partial_\infty G$

NOTE $\Sigma_2 \hookrightarrow K$ AND ALSO

$S^1 \cong \partial_\infty(\pi_1 \Sigma_2) \hookrightarrow \partial_\infty \pi_1(K)$ ...