GEOMETRIC GROUP THEORY
MA4H4 FALL 2023

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1. Introduction

Groups are algebraic objects, consisting of

- a set G and
- an operation $G \times G \to G$, written $(a, b) \mapsto a \cdot b$

satisfying three axioms:

(1) There is an *identity* element $e \in G$ satisfying $e \cdot a = a \cdot e = a$ for all $a \in G$.

(2) Every element $a \in G$ has an *inverse* a^{-1} satisfing $a \cdot a^{-1} = a^{-1} \cdot a = e$.

(3) Elements satisfy the *associative law*, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.

But this algebraic definition hides the fact that groups are closely tied to geometry. The basic observation is that the symmetries of a geometric object form a group, with the operation of composition.

The goal of geometric group theory is to understand a given group G. The method is to

- find a geometric object X on which G acts as symmetries, then
- study the geometry and topology of X to learn about algebraic properties of G.

This idea is as old as the definition of groups, but has become more and more developed and powerful as time has passed. Here are some of the key figures in the history of geometric group theory:

1.1. **Evariste Galois (1832).**

Galois introduced the notion of a *group* while studying field extensions. The groups he studied are now called *Galois groups*.

1.2. **Felix Klein (1872).**

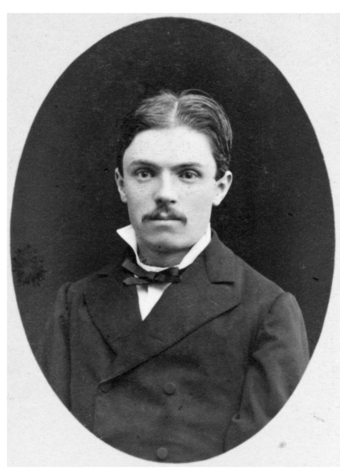
Klein had the opposite goal from what we stated above: he wanted to understand geometric spaces (Euclidean spaces, projective spaces, hyperbolic spaces, etc.) by using algebra to study their symmetry groups. This is known as Klein's Erlangen program. It helped to establish the deep connections between geometry and group theory.

The symmetry groups Klein studied were continuous groups, in fact the groups are themselves manifolds (i.e. they are Lie groups).

However, the main focus of geometric group theory is on *discrete groups*. This means they have the discrete topology, i.e. every element is both open and closed.

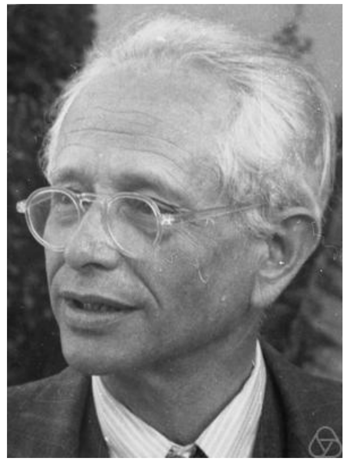
Example 1.1. *the real numbers $\mathbb{R}$ form a Lie group under addition. The subgroup $\mathbb{Z}$ of integers is discrete. The group $\mathbb{Z}$ can be thought of as a discrete subset of the line or as a group of translations of the line.*

1.3. **Henri Poincaré 1895.**



Poincaré defined the fundamental group of a topological space, showed the universal covering space of a closed surface can be identified with the hyperbolic plane, and identified the fundamental group of the surface with the deck transformations.

1.4. **Max Dehn 1911.**



Dehn studied groups by looking at generators and relations between the generators. He asked three algorithmic questions that are still basic questions in geometric group theory. Namely,

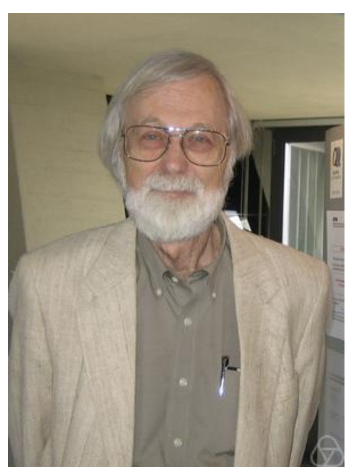
(1) The *Word Problem*: Is there an algorithm to decide whether or not a product of generators is the identity in the group?

(2) The *Conjugacy Problem*: Is there an algorithm to decide whether or not two words in the generators are conjugate in the group?

(3) The *Isomorphism Problem*: Is there an algorithm to decide whether or not two groups given by generators and relations are isomorphic?

In 1912 Dehn gave algorithms that solve these problems if you know your groups are fundamental groups of surfaces. He did this by realizing the surface group as symmetries of the hyperbolic plane, then using hyperbolic geometry.

1.5. **Albert Švarc 1955, John Milnor 1968.**

These two independently studied what are now known as *quasi-isometries* between metric spaces; these are maps that preserve the metric approximately, but not exactly. They proved the Švarc-Milnor lemma, which is sometimes called the *Fundamental Theorem of Geometric Group Theory*.

A group can be made into a metric space, by choosing a generating set, then defining the distance between a and b to be the minimal length of the element $a^{-1}b$ as a word in those generators. The Švarc-Milnor lemma says that a metric space with a sufficiently nice group action is quasi-isometric to the group itself.

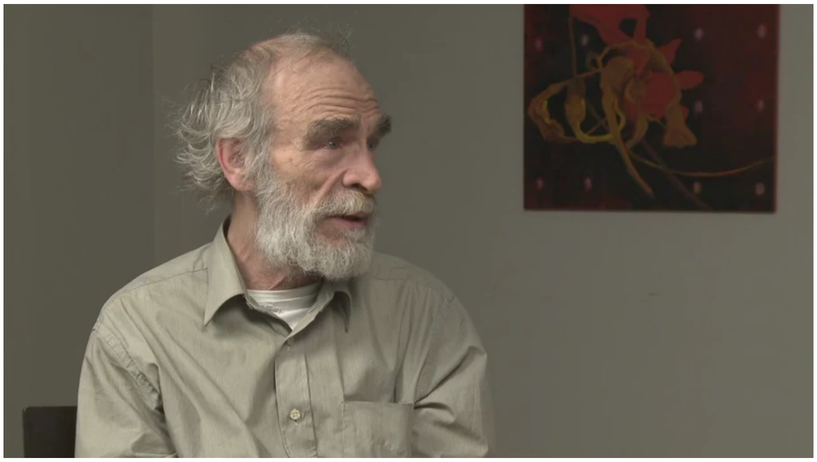
1.6. **John Stallings 1982.**

Stallings introduced ingenious topological methods for the study of free groups and their automorphisms.

1.7. **William P. Thurston 1970's.**

Thurston studied 3-manifolds by studying their fundamental groups and their action by deck transformations on their universal covers. He conjectured a complete classification of 3-manifolds according to the geometry of their universal covers. This classification was proved to be correct by Perelman in 2000. It included a solution to a famous conjecture of Poincaré, which says that the only closed oriented 3-manifold homotopy equivalent to the 3-sphere is the 3-sphere itself.

1.8. **Misha Gromov 1987.**



Gromov is primarily a geometer, who promoted the idea that one should consider groups as metric spaces, using the word-length metric described above. He injected a large number of geometric ideas into the study of finitely-generated groups, for example he defined notions of negative and non-positive curvature that make sense for groups. He proved in particular that negatively curved groups (now called *Gromov hyperbolic groups*) have many strong algebraic properties. Although geometric group theory has historical roots in all of the work mentioned above, its emergence as a distinct field of mathematics can be attributed to Gromov's work.

2. Course topics

- Free groups and ping-pong
- Brief review of fundamental groups and covering spaces
- Cayley graphs
- Group presentations and presentation complexes
- Quasi-isometries
- The Švarc-Milnor Lemma
- Brief review of the hyperbolic plane
- Surface groups
- $SL(2,\mathbb{Z})$.
- Definition and examples of Gromov hyperbolic groups
- Properties of Gromov hyperbolic groups
- Definition and examples of CAT(0) groups
- Properties of CAT(0) groups

Generators, $\mathbb{Z}^n$, $SL(n,\mathbb{Z})$, free constructions

Let G be a group.

A subset $S \subset G$ generates G if every $g \in G$ can be written as a product of elements of S and their inverses

G is finitely generated if some finite $S \subset G$ generates G

$\boxed{\text{We focus on finitely generated infinite groups } G}$

Simplest example: $\mathbb{Z}$, generated by $\{1\}$

Next: $\mathbb{Z}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z}\}$

generated by $\{e_i = (0,\ldots,0,\underset{\text{ith coordinate}}{1},0,\ldots,0)\}_{i=1,\ldots,n}$

$\mathbb{Z}^n$ is abelian: $gh = hg$ for all g, h

In algebra you probably classified finitely-generated abelian groups:

$$G \cong \mathbb{Z}^m \oplus \underbrace{\mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{n_k}\mathbb{Z}}_{\text{finite order elements, the torsion subgroup}}$$

$m, n_i \in \mathbb{N}$

Important non-abelian example:

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$$

Generated by $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

$\boxed{\text{Freeness}}$

$\mathbb{Z}^n$ is called the <u>free abelian group</u> of rank n.
Why free?

$e_i + (-e_i)$ $e_i - e_i = 0$ (group) $e_i + e_j = e_j + e_i$ (abelian) but there are no <u>other relations</u> between generators.

<u>Example of a relation in an abelian group</u>:
$\mathbb{Z}/2 \oplus \mathbb{Z}/3$ is generated by $e_1 = (1,0)$ and $e_2 = (0,1)$
$2(1,0) + 3(0,1) = (0,0)$ is a non-trivial relation
$2e_1 + 3e_2 = \cancel{id}\ 0$ $2e_1 = 0$ $3e_2 = 0$

In $\mathbb{Z}^n$: If $\sum n_i e_i = 0$ then $n_i = 0$ for all i

<u>Conclusion</u>: A f.g. abelian group is free abelian if and only if it has no torsion.

<u>Exercise</u>: Why does free abelian group have no torsion?

A formal (and generalizable) way to say "free abelian"

Let $S = \{e_1, \dots, e_n\}$

Any set map $S \xrightarrow{f} G$ to an abelian group G

Extends uniquely to a group homomorphism

$$\mathbb{Z}^n \xrightarrow{\hat{f}} G$$

Picture:

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[dr, dashed, "\exists ! \hat{f}"] & \\
S \arrow[u, phantom, "\cup" description] \arrow[r, "f"'] & G
\end{tikzcd} \qquad (*)$$

This is called a "universal property" in the category of abelian groups

why does this say there are no relations?

Suppose $r = n_1 e_1 + \dots + n_k e_k = \vec{0}$, $n_i \neq 0$

Let $f_i : S \longrightarrow \mathbb{Z}$

$e_i \longmapsto 1$

$e_j \longmapsto 0$, $j \neq i$

This extends to a ! homomorphism by the universal property

$$\mathbb{Z}^n \xrightarrow{f_i} \mathbb{Z}.$$

$$f(n_1 e_1 + \dots + n_k e_k) = n_1 f(e_1) + \dots + n_k f(e_k) = n_i$$

but any homomorphism must send $0 \longrightarrow 0$, so $n_i = 0$.

Now let's drop "abelian" and define <u>free</u> groups:

F is <u>free</u> if there is $S \subset F$ such that

- any set map $S \xrightarrow{f} G$ to a group G extends uniquely to a homomorphism $F \xrightarrow{\hat{f}} G$.

<u>picture</u>

$$\begin{tikzcd} F \arrow[drr, dashed, "\exists!\,\hat{f}"] & & \\ S \arrow[u, phantom, "\cup" description] \arrow[rr, "f"] & & G \end{tikzcd} \qquad (*)$$

We say F is free on S, write $F = F\langle S\rangle$,
or S is a basis for F.

<u>Questions</u>
(1) If we can find such an S, does it generate F?
(2) Can F be free on a different set S'?
(3) If so what is the relation between S and S'?
(4) Why does this say there are no relations?
(5) Is there a free group?

Observations: (Consequences of the universal property):
(0) Suppose F is free on $S \subset F$.
The identity $F \to F$ extends the inclusion
so is the <u>unique</u> homomorphism extending the inclusion, by the universal property.

$$\begin{tikzcd} F \arrow[dr, "id"] & \\ S \arrow[u, phantom, "\cup" description] \arrow[r, hook, "i"] & F \end{tikzcd}$$

(1) S generates F; ie every elt of F is a product of elts of $S \cup S^{-1}$.

Pf: The set of elts that are products of $S \cup S^{-1}$ is a subgroup G (it contains inverses, and is closed under $\cdot$) multiplication

so

$$\begin{tikzcd} F \arrow[drr, dashed, "\hat{i}"] & & \\ \cup & & \\ S \arrow[rr, hook, "i"] & & G \subsetneq F \end{tikzcd}$$

$\exists!\ \hat{i} : F \to G$

If $G \subsetneq F$, the composition $\begin{tikzcd} F \arrow[r, "\hat{i}"] \arrow[rr, bend left, "h"] & G \arrow[r, phantom, "\subsetneq"] & F \end{tikzcd}$ extends $S \subseteq F$ but is not surjective, so is not the identity, contradicting point (0) above.

(2)-(3) Suppose F is free on S and on S' (both countable. Then $|S| = |S'|$

Proof:

Any homomorphism $\underset{\cup 1}{F} \longrightarrow \mathbb{Z}/2$ restricts to a unique set map $S \longrightarrow \mathbb{Z}/2$

Conversely, any set map $S \longrightarrow \mathbb{Z}/2$ extends to a unique homomorphism $F(S) \longrightarrow \mathbb{Z}/2$

$$\boxed{\text{ie } (\mathrm{Hom}(F, \mathbb{Z}/2)) \leftrightarrow (\text{maps } S \to \{0,1\})}$$

Now count: there are $2^{|S|}$ set maps $S \to \{0,1\}$

so $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S|}$

Similarly, $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S'|}$

$\therefore\ 2^{|S|} = 2^{|S'|}$ so $|S| = |S'|$ ✓

Definition: $|S|$ is the rank of $F\langle S \rangle$

Theorem Two finitely-generated free groups are isomorphic if and only if they have the same rank.

proof $\Rightarrow$ Let F, F' be finitely-generated free groups, $F = F(S)$

and $f : F \to F'$ an isomorphism

$s \mapsto f(s)$

Claim F' is free on $S' = f(S)$.

pf We verify the universal property.

Given

$$\begin{tikzcd} f(S) \arrow[r, "g"] \arrow[d, phantom, "\cup"] & G \\ F' \arrow[ur, dashed, "\hat{g}"'] & \end{tikzcd}$$

a set map, we need a unique homomorphism $\hat{g} : F' \to G$ extending g.

existence

Define g by: $\hat{g}(y) = \widehat{gf}(f^{-1}(y))$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, phantom, "\cap"] & f(S) \arrow[r, "g"] \arrow[d, phantom, "\cap"] & G \\ F \arrow[r, "f"] \arrow[urr, "\widehat{gf}"] & F' \arrow[ur, dashed] & \\ f^{-1}(y) & y \arrow[l, mapsto] & \end{tikzcd}$$

This extends g: if $y = f(s)$, then

$$\widehat{gf}(f^{-1}(s)) = g \circ f \circ f^{-1}(s) = g(s) \quad \checkmark$$

<u>uniqueness</u> follows from the uniqueness of f^{-1} and $\widehat{gf}$.

$\Leftarrow$ Suppose $|S| = |S'|$. Choose a bijection $b : S \to S'$

$$\begin{tikzcd} F(S) \arrow[r] & F(S') \\ S \arrow[u, phantom, "\cup\, i"] \arrow[r, "b"] & S' \arrow[u, phantom, "\cup\, i'"] \end{tikzcd} \quad \text{extends } i' \circ b$$

$$\begin{tikzcd} F(S') \arrow[r] & F(S) \\ S' \arrow[u, phantom, "i'\, \cup"] \arrow[r, "b^{-1}"] & S \arrow[u, phantom, "\cup\, i"] \end{tikzcd} \quad \text{extends } i \circ b^{-1}$$

The composition
$$\begin{tikzcd} F(S) \arrow[r, "\widehat{i' \circ b}"] & F(S') \arrow[r, "\widehat{i \circ b^{-1}}"] & F(S) \\ S \arrow[u, phantom, "\cup"] \arrow[r, "b"] & S' \arrow[u, phantom, "\cup"] \arrow[r, "b^{-1}"] & S \arrow[u, phantom, "\cup"] \end{tikzcd}$$

extends id_S, so $= id_{F(S)}$ by (0) $\checkmark$

(4) Why does the universal property imply there are no relations other than $gg^{-1} = e$?

Let $S = \{s_1, s_2, \ldots\}$ and suppose $w = e$ is a non-trivial relation in F, ie

$$w = a_1^{n_1} \cdots a_k^{n_k}, \quad a_i \in S,$$

$$a_{i+1} \neq a_i, \quad n \in \mathbb{Z} \setminus \{0\}.$$

If $k = 1$, ie $w = s^n$, $s \in S$

Define a set map

$$\begin{aligned} S &\xrightarrow{f} \mathbb{Z} \\ s &\longmapsto 1 \\ s' &\longmapsto 0 \text{ if } s' \neq s \end{aligned}$$

extend to a homomorphism $F \xrightarrow{\hat{f}} \mathbb{Z}$

$$\begin{tikzcd} F \arrow[dr, "\hat{f}"] & \\ S \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "f"] & \mathbb{Z} \end{tikzcd}$$

by the universal property

Since $\hat{f}$ is a homomorphism, $f(e) = 0$

ie $\hat{f}(w) = 0$

but $\hat{f}(w) = \hat{f}(s^n) = n \neq 0$ $\times$.

This doesn't work if w involves 2 or more elements of S! Here is an argument that works in general:

(Proving the existence of F_2)

Define $f : S \longrightarrow SL(2, \mathbb{Z})$ by

$$a_i \longmapsto B^i A B^{-i},$$

where $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$.

and extend to a homomorphism

$$\hat{f} : F(S) \longrightarrow SL(2, \mathbb{Z})$$

$\hat{f}$ is a homomorphism, so $\hat{f}(e) = \hat{f}(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

But $\hat{f}(w) = B^i A^{n_1} \underbrace{B^{-i} B^{j}}_{i \neq j} A^{n_2} B^{-j} \cdots B^{l} A^{n_k} B^{-l}$

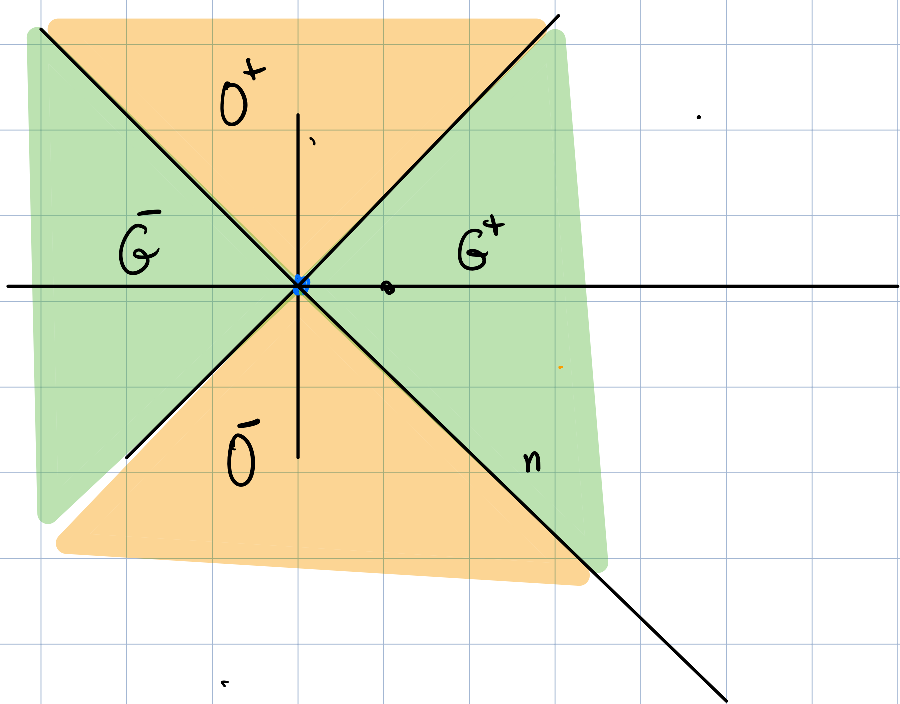
ie $\hat{f}(w)$ alternates between non-trivial powers of B and of A

Claim No such product is the identity $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ in $SL(2,\mathbb{Z})$

Proof $A^k\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&2k\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+2ky\\y\end{pmatrix}$

$B^k\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&0\\2k&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\2kx+y\end{pmatrix}$

Let $G = \{(x,y) \mid |x|>|y|\}$, $O = \{(x,y) \mid |y|> x|\}$



Claim $A^k(O) \subseteq G$, $B^k(G) \subseteq O$

Pf $|y|>|x| \Rightarrow |x+2ky| > |y| \Rightarrow A^k\begin{pmatrix}x\\y\end{pmatrix} \in G$

$|x|>|y| \Rightarrow |y+2kx| > |x| \Rightarrow B^k\begin{pmatrix}x\\y\end{pmatrix} \in O$ ✓

$\hat{f}(w)$ ends in a power of B, say B^i.

If $i > 0$, $B^i\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\3i+1\end{pmatrix} \in$ Orange

Since $A^{\pm n}(\text{Orange}) \subseteq$ Green

and $B^{\pm n}(\text{Green}) \subseteq$ Orange, the

image of $\begin{pmatrix}1\\1\end{pmatrix}$ bounces between Orange and Green, so

never returns to the diagonal, in particular to $\begin{pmatrix}1\\1\end{pmatrix}$.

so $\dots B^{m_2} A^{n_2} B^{m_1} A^{n_1} B^i \neq \begin{pmatrix}1&0\\0&1\end{pmatrix}$

If $i<0$, $B^i\begin{pmatrix}-1\\1\end{pmatrix} \in$ Orange, so a similar

argument shows $\dots B^{m_2} A^{n_2} B^{m_1} A^{n_1} B^i \neq \begin{pmatrix}1&0\\0&1\end{pmatrix}$

⑤ Is there a free group?

Let S be a set, and $\bar{S}$ another copy

ie $S \longleftrightarrow \bar{S}$ a bijection.

$a \longleftrightarrow \bar{a}$

Let $A = S \cup \bar{S}$ (A for "alphabet")

we have an involution $A \to A$

$a \to \bar{a}$

$\bar{a} \to a$

I will construct a group F containing S, then prove it is free.

A word in A is a finite string of elements of A $a_1 \ldots a_k$

A word is reduced if $a_{i+1} \neq \bar{a}_i$ for all i.

The elements of F are the reduced words in A plus the empty word $\varnothing$

The operation $F \times F \longrightarrow F$

is "juxtaposition followed by reduction"

ie

if $X = x_1 \cdots x_k$ $\quad Y = y_1 \cdots y_\ell$ reduced words

Let
$$r(X,Y) = \begin{cases} 0 & \text{if } y_1 \neq \bar{x}_k \\ \max\{j : y_i = \bar{x}_{k-i+1} \ \forall\, i \leq j\} & \text{otherwise} \end{cases}$$

(eg $\quad x_1 \cdots \cancel{x_k}\, \cancel{y_1} \cdots y_\ell \qquad r=1$)

Define
$$XY = x_1 \cdots x_{k-r}\, y_{r+1} \cdots y_\ell$$
this is reduced so is in F.

<u>Claim</u> F with this product is a group

① identity $= \phi$

② inverses: $(x_1 \cdots x_k)^{-1} = \bar{x}_k \cdots \bar{x}_1$

③ associative law

$$\left.\begin{aligned} X &= x_1 \cdots x_k \\ Y &= y_1 \cdots y_\ell \\ Z &= z_1 \cdots z_m \end{aligned}\right\} \quad (XY)Z \overset{?}{=} X(YZ)$$

Check: let $r = r(x, y)$
$s = r(y, z)$

Look at $$\underbrace{x_1 \cdots x_k}_{r}\ \underbrace{y_1 \cdots}_{r}\ \underbrace{\cdots y_\ell}_{s}\ \underbrace{z_1 \cdots}_{s} \cdots z_m$$

Suppose first $r+s < \ell$:

$$\overbrace{x_1 \cdots}^{a}\ \underbrace{\overbrace{\cancel{x_k}}^{u}}_{r}\ \underbrace{\overbrace{\cancel{y_1}}^{\bar u}}_{r}\ \overbrace{\cdots}^{b}\ \underbrace{\overbrace{y_\ell}^{v}}_{s}\ \underbrace{\overbrace{z_1}^{\bar v}}_{s}\ \overbrace{\cdots z_m}^{c}$$

$XY = (au)(\bar u b v) = abv \qquad (XY)Z = (abv)(\bar v c) = abc$

$YZ = (\bar u b v)(\bar v c) = \bar u b c \qquad X(YZ) = (au)(\bar u b c) = abc$

abc is reduced, so these are equal in F ✓

If $r+s = \ell$ ($b = \emptyset$ in the above picture)

$$\overbrace{x_1 \cdots}^{a}\ \underbrace{\overbrace{\cancel{x_k}}^{u}}_{r}\ \underbrace{\overbrace{\cancel{y_1}}^{\bar u}}_{r}\ \underbrace{\overbrace{y_\ell}^{v}}_{s}\ \underbrace{\overbrace{z_1}^{\bar v}}_{s}\ \overbrace{\cdots z_m}^{c}$$

$XY = (au)(\bar u v) = av \Rightarrow (XY)Z = (av)\bar v c = ac$

and $YZ = (\bar u v)(\bar v c) = \bar u c \Rightarrow X(YZ) = (au)(\bar u c) = ac$

ac may not be reduced, but a and c are so $ac \in F$ is well-defined

If $r+s > \ell$

$$\underbrace{\overbrace{x_1 \cdots}^{a}\overbrace{\cdots x_k}^{\bar v \bar u}}_{r}\ \underbrace{\overbrace{y_1 \cdots}^{u}\overbrace{\cdots}^{v}}_{r}\underbrace{\overbrace{\cdots y_\ell}^{w}}_{s}\ \underbrace{\overbrace{z_1 \cdots}^{\bar w \bar v}}_{s}\overbrace{\cdots z_m}^{c}$$

$$a\,\underbrace{\bar v \bar u}\,\underbrace{u v w}\,\bar w \bar v c$$

$XY = (a\bar v \bar u)(uvw) = aw$

$(XY)Z = (aw)(\bar w \bar v c) = a\bar v c$

$YZ = (uvw)(\bar w \bar v c) = uc$

$X(YZ) = (a\bar v\bar u)(uc) = a\bar v c$ ✓

so associativity holds, and we have a group.
Call it $F(S)$

Does $F(S)$ satisfy $(*)$?
A: Yes: Let $a_1^{n_1} \cdots a_k^{n_k} \in F(S)$

Given $S \xrightarrow{\varphi} G$ a set map
$a_i \longmapsto g_i$

define a homomorphism $f: F(S) \to G$
by $f(a_1^{n_1} \cdots a_k^{n_k}) = g_1^{n_1} \cdots g_k^{n_k}$.

Since any homomorphism must do this,
f is uniquely defined.

Free products

G_1, G_2 groups. The free product $G_1 * G_2$ is ~~a~~ the group with the following universal property:

$\exists$ inclusions $G_1 \xhookrightarrow{i_1} G_1 * G_2$ and $G_2 \xhookrightarrow{i_2} G_1 * G_2$ such that: given a group K and homomorphisms $f_i : G_i \longrightarrow K$

$\exists!$ homomorphism $f : G_1 * G_2 \longrightarrow K$ s.t. $f i_2 = f_2$
$f i_1 = f_1$.

Picture

(*)

$$\begin{tikzcd}
G_1 \arrow[r, "i_1"] \arrow[dr, "f_1"'] & G_1 * G_2 \arrow[d, dashed, "\exists! f"] & G_2 \arrow[l, hook', "i_2"'] \arrow[dl, "f_2"] \\
& K &
\end{tikzcd}$$

Exercise: Prove uniqueness.

The free product $G * H$ can be constructed using words, like the free group:
The alphabet is $A = (G \setminus e_G) \sqcup (H \setminus e_G)$

A word $x_1 \cdots x_k$ is <u>reduced</u> if $x_i \in G \Rightarrow x_{i+1} \in H$
and $x_i \in H \Rightarrow x_{i+1} \in G$

W = set of reduced words (including $\emptyset$)

Operation $W \times W \longrightarrow W$ defined inductively on $k+l$.

$$(x_1 \cdots x_k, y_1 \cdots y_l) = \begin{cases} x_1 \cdots x_{k-1} y_2 \cdots y_l & \text{if } y_1 = x_k^{-1} \in G \text{ or } H \\ x_1 \cdots x_{k-1}(x_k y_1) y_2 \cdots y_l & \text{if } x_k, y_1 \in G \text{ or } H \\ & \quad y_1 \neq x_k^{-1} \\ x_1 \cdots x_k y_1 \cdots y_l & \text{if } x_k \in G, y_1 \in H \\ & \text{or } x_k \in H, y_1 \in G \end{cases}$$

ie "juxtaposition followed by multiplication and cancellation."

Identity $\emptyset$ Associative ✓

Inverse ✓

eg $G_1 = G_2 = \mathbb{Z}$ $\quad G_1 = \langle t \rangle \; G_2 = \langle s \rangle \quad (t^n \leftrightarrow n \in \mathbb{Z})$

$W = t^{n_1} s^{m_1} t^{n_2} \cdots t^{n_k} s^{m_k}, \quad n_i, m_i \neq 0$

[could start or end with either an s or a t]

Then $G_1 * G_2 \cong F\langle s, t \rangle$

Free products with amalgamation:

Given groups G_1, G_2, H and homomorphisms $h_i : H \longrightarrow G_i$

$G_1 *_H G_2$ is a (the) group satisfying the following universal property:

$\exists$ $i_1 : G_1 \longrightarrow G_1 *_H G_2$ and $i_2 : G_2 \longrightarrow G_1 *_H G_2$
such that: For any group K and homomorphisms

$$g_1 : G_1 \longrightarrow K$$
$$g_2 : G_2 \longrightarrow K$$

that agree on H, (ie $g_1 h_1(y) = g_2 h_2(y)$ for all $y \in H$)

$\exists!$ $g : G_1 *_H G_2 \longrightarrow K$ s.t. $g i_1 = g_1$ and $g i_2 = g_2$.

<u>Picture</u>

(*)
$$\begin{tikzcd}
H \arrow[r, "h_1"] \arrow[d, "h_2"'] & G_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "g_1"] & \\
G_2 \arrow[r, "i_2"] \arrow[drr, bend right, "g_2"'] & G_1 *_H G_2 \arrow[dr, dashed, "\exists! g"] & \\
& & K
\end{tikzcd}$$

Called the <u>free product with amalgamation</u>.

To show existence:

<u>Claim</u> Let N = subgroup of $G_1 * G_2$ normally generated by words $i_1 h_1(y)(i_2 h_2(y))^{-1}$. Then $G_1 * G_2 / N$ has the universal property we want.

ie
$$\begin{tikzcd} H \arrow[r, "h_1"] \arrow[d, "h_2"'] & G_1 \arrow[d] \arrow[rdd, bend left, "g_1"] & \\ G_2 \arrow[r] \arrow[rrd, bend right, "g_2"'] & G_1 * G_2 / N \arrow[rd, dashed, "\exists! g"] & \\ & & K \end{tikzcd}$$

Define $g(\underline{x_1 x_2 \cdots x_k N}) = g_1(x_1) g_2(x_2) \cdots g_{1 \text{ or } 2}(x_k)$

$(x_i \in G_1 \Rightarrow x_{i+1} \in G_2,\ x_i \in G_2 \Rightarrow x_{i+1} \in G_1)$

Have to show $g(N) = e_K$.

N is normally gen by elements $h_1(y) h_2(y)^{-1}$, $y \in H$

We know $y \in H \Rightarrow g_1 h_1(y) = g_2 h_2(y)$

so $g(\underbrace{h_1(y)}_{\in G_1} \underbrace{h_2(y)^{-1}}_{\in G_2}) = g_1(h_1(y)) g_2(h_2(y)^{-1})$

$= g_1(h_1(y)) g_2(h_2(y))^{-1} = e_K$

For any $\underline{\text{conjugate}}$ of y, also

$(x y x^{-1}) = g(x) \cdot e_K\, g(x)^{-1} = e_K$

So g is well-defined on $G_1 * G_2 / N$.

(Still have to show g is unique, but that is straight forward.)

Group actions:

We want a connection between groups G and spaces X

If X is a topological space, the set of homeomorphisms $X \to X$ forms a group $\mathrm{Homeo}(X)$, with composition as its operation.

If X is a metric space, the set of isometries forms a group $\mathrm{Isom}(X)$.

If X is a smooth manifold, the set of diffeomorphisms forms a group $\mathrm{Diff}(X)$.

Other structures (symplectic, Kähler, complex...) give other groups of symmetries preserving the structure.

Let $\mathrm{Symm}(X)$ denote one of these groups.

__Def__ A (left) action of G on X is a homomorphism $G \xrightarrow{\rho} \mathrm{Symm}(X)$, ie
$\rho(gh) = \rho(g) \circ \rho(h)$ "first apply h, then g"

__ie:__ For each $g \in G$ there is a symmetry $X \longrightarrow X$, written $x \longmapsto g \cdot x$
such that $gh \cdot x = g(h \cdot x)$ __and__ $e_G \cdot x = x$.

(A right action of G on X is a map
$p: G \longrightarrow \mathrm{Sym}(X)$ such that
$p(gh) = p(h) \circ p(g)$ and $p(e_G) = id_X$

ie p is an "anti-homomorphism"

ie for every $g \in G$, there is a symmetry $X \to X$
written $x \mapsto x \cdot g$
satisfying $x \cdot e_G = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$
"first apply g, then h")

Suppose $p: G \longrightarrow \mathrm{Sym}(X)$ is a left action.

Let $x \in X$. The stabilizer $G_x = \{ g \in G \mid gx = x \}$
(This is a subgroup).

The action is free if $G_x = \{e_G\}$ for all x
ie every point of X is "moved" by every group element.

The action is faithful if $p: G \longrightarrow \mathrm{Sym}(X)$ is injective

ie $g \cdot x = x$ for <u>all</u> $x \in X \Rightarrow g = e_G$.
"the only group element that fixes every $x \in X$ is the identity"

<u>Exercise</u>: If the action of G on X is <u>not</u> faithful there is an induced action of $G/\ker(\rho)$ on X which <u>is</u> faithful.

<u>Notation</u> a left action of G on X will often be denoted $G \curvearrowright X$.

An action $G \curvearrowright X$ is proper if for every compact set $K \subset X$, $\{g \mid gK \cap K \neq \emptyset\}$ is finite.

If $G \curvearrowright X$ and $x \in X$, the orbit of x is $Gx = \{gx \mid g \in G\}$

An action $G \curvearrowright X$ is cocompact if there is a compact $K \subset X$ whose translates cover X:

$$X = \bigcup_{g \in G} gK$$

If $G \curvearrowright X$, define an equivalence relation on X by $x \sim x'$ if $x' \in Gx$.

The orbit space (or quotient space) $G \backslash X$ is the set of equivalence classes, with the quotient topology

($U \subset G\backslash X$ is open $\Leftrightarrow$ $p^{-1}(U)$ is open in X, where $p: X \longrightarrow G\backslash X$
$x \longmapsto G \cdot x$.)

<u>Exercise</u> If the action is cocompact, then the quotient space is compact.

aside:

$\boxed{\text{Actions by isometries on metric spaces}}$

(reference: M. Kapovich, A note on properly discontinuous actions, arXiv: 2301.05325)

Saying $G \backslash X$ is compact is a common way to define "cocompact action."
If X is a metric space and G acts by isometries, then this is equivalent to our definition.

If X is a metric space and G acts by isometries, "proper" is equivalent to:

every $x \in X$ has a neighborhood U_x such that $\{ g \mid g U_x \cap U_x \neq \emptyset \}$ is finite.

Our definition of proper $\Rightarrow$ G_x is finite for all $x \in X$.

Example $\mathbb{Z} \curvearrowright \mathbb{R}$ by translations

$n \mapsto (x \mapsto x+n)$

is
- free $(x+n=x \Rightarrow n=0$, ie $\mathbb{Z}_x = 0\ \forall x)$
- faithful $(n \neq 0 \Rightarrow x+n \neq x)$
- proper $(K \subset \mathbb{R} \Rightarrow K \subset [a,b]$ for some $a,b \in \mathbb{Z}$, $K+n \cap K \neq \emptyset \Rightarrow n \leq (b-a))$
- cocompact $\left(\bigcup_{n \in \mathbb{Z}} [0,1]+n = \mathbb{R}\right)$

The quotient space $\mathbb{Z} \backslash \mathbb{R}$ is homeomorphic to S^1.

Example $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle s \rangle * \langle t \rangle$
= infinite dihedral group

acts on $\mathbb{R}$ by
$s: x \longmapsto -x$
$t: x \longmapsto 1-x$

$(ts: x \xrightarrow{s} -x \xrightarrow{t} 1-(-x) = 1+x.)$

not free, but proper, cocompact, faithful

Example $SL(2,\mathbb{Z})$ acting on $\mathbb{R}^2$ by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

faithful $\left(A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}\ \forall \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

not free $(\operatorname{stab}(0,0) = SL_2\mathbb{Z}\,!)$

not proper $(A \cdot B_N \cap B_N \ni (0,0)\ \forall A)$

not cocompact (exercise)

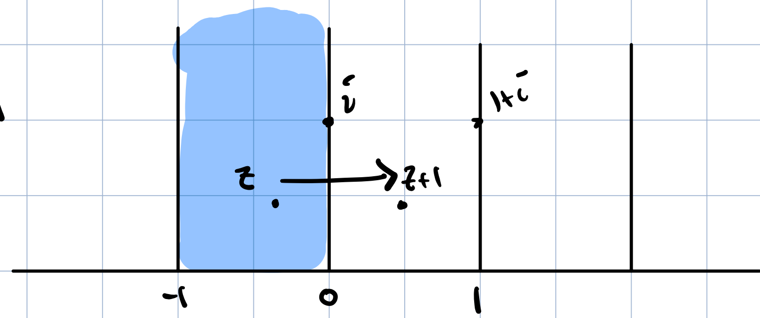
$SL(2,\mathbb{Z})$ also has a very interesting action on a different space X:

$$X = \mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

eg $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot z = z+1$

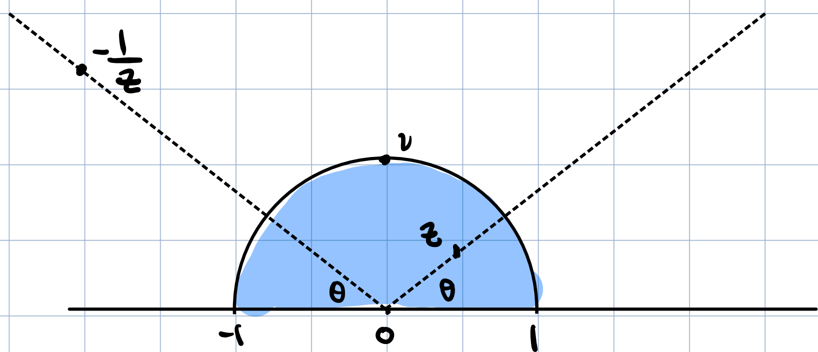
(translation right by 1)



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = \frac{-1}{z} = \frac{-\bar{z}}{z\bar{z}}$$

$$re^{i\theta} \longmapsto \frac{1}{r} e^{i(\pi-\theta)}$$

(inversion in the circle $|z|=1$ followed by reflection in y-axis)



This action is not free: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot i = \frac{-1}{i} = i$

and not faithful: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = \frac{-1}{-z} = z$

(but $\ker = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $PSL_2\mathbb{Z} = SL_2(\mathbb{Z}) / \pm I$ does act faithfully.)

and not cocompact: K compact $\Rightarrow K \subset B_r$ for some r. I claim there is no $A \in SL(2,\mathbb{Z})$ s.t. $A \cdot B_r \ni \begin{pmatrix} n \\ 0 \end{pmatrix}$ for $n > r$.

(Since translates of B_r don't cover $\mathbb{R}^2$, neither do translates of K)

proof: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n \\ 0 \end{pmatrix}$

$$\Rightarrow \quad \begin{aligned} ax + by &= n \\ cx + dy &= 0 \end{aligned} \quad \Rightarrow d = 0 \text{ or } y = \frac{-c}{d}x$$

$$y = -\frac{c}{d}x \Rightarrow ax - b\frac{c}{d}x$$

$$\Rightarrow (ad - bc)x = dn$$

$$\Rightarrow x = dn, \ y = -cn$$

$$\Rightarrow x^2 + y^2 = n^2(d^2 + c^2)$$

$$\geq n^2 > r^2 \checkmark$$

$$d = 0 \Rightarrow -bc = 1 \Rightarrow b = \pm c = \pm 1$$

$$by = n \Rightarrow y = \pm n, \ x = 0$$

$$\Rightarrow x^2 + y^2 = n^2 > r^2 \checkmark$$

But: it is proper:

In particular $G_x = \mathbb{Z}/2\mathbb{Z}$ unless x is in the orbit of i or $\omega = \frac{1+\sqrt{3}i}{2}$.

and $G_i \cong \mathbb{Z}/4\mathbb{Z}$, $G_\omega \cong \mathbb{Z}/6\mathbb{Z}$

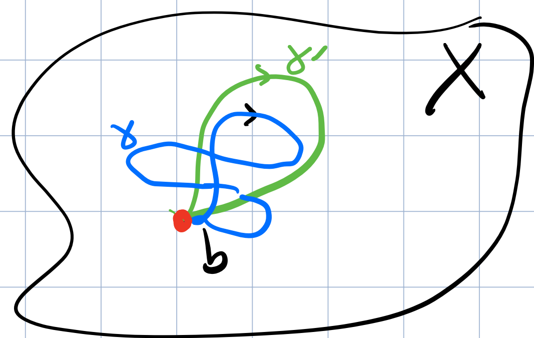
Basic example of a free and proper action: $\pi_1(X,b)$ acting on $\tilde{X}$ by deck transformations.

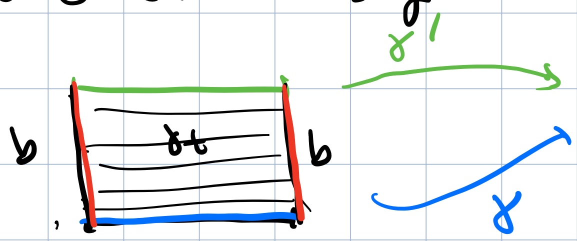
Reference: Hatcher, Algebraic Topology, Chapter 1.

Following is a brief review of what you need to know. This should all be familiar.

If X is a topological space and $b \in X$, the fundamental group $\pi_1(X,b)$ is the group of equivalence classes of loops, where a loop is a continuous map $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = \gamma(1) = b$.

$\gamma \sim \gamma'$ if γ' can be deformed to γ continuously

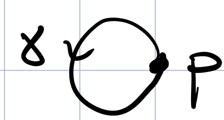




The operation: $\gamma_1 . \gamma_2$ means first do γ_1, then γ_2 both at double speed. Check $[\gamma_1].[\gamma_2] = [\gamma_1 . \gamma_2]$. we will usually omit the notation $[\;]$.

This is a group; the identity is the constant map $\gamma(t) = b$ for all $t \in [0,1]$.

__Example__. X contractible $\Rightarrow$ $\pi_1(X,b) = \langle 1 \rangle$

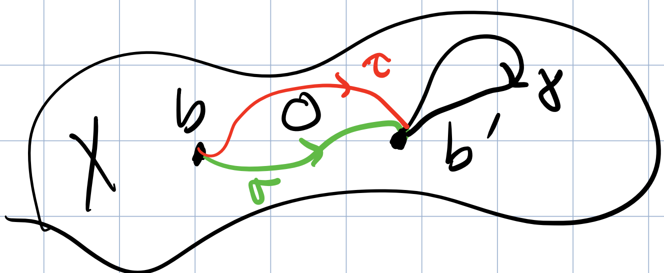
__Example__ $\pi_1(S^1, p) \cong \mathbb{Z} = \langle \gamma \rangle$, 

If X is path-connected there is an isomorphism $\pi_1(X, b') \xrightarrow{\cong} \pi_1(X, b)$ for any $b, b' \in X$ defined as follows:

Choose a path $\sigma : [0,1] \to X$ with $\sigma(0) = b$, $\sigma(1) = b'$

Define $\sigma_* : \pi_1(X, b') \to \pi_1(X, b)$ by

$$\sigma_*(\gamma) = \sigma \cdot \gamma \cdot \sigma^{-1} \text{ (triple speed)}$$



Choosing a different path τ gives a different isomorphism.

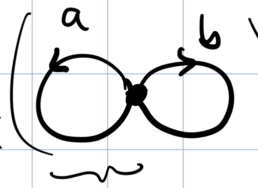
The composition $\pi_1(X, b') \xrightarrow{\sigma_*} \pi_1(X, b) \xrightarrow{\tau_*^{-1}} \pi_1(X, b')$

is __an inner automorphism__ (conjugation by the loop $\sigma^{-1} \cdot \tau$.)

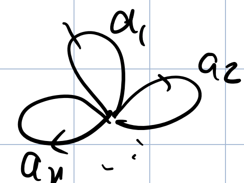
__Van Kampen's Theorem__ If $X = A \cup B$ with A, B open and A, B and $A \cap B$ path-connected, and $p \in A \cap B$ then

$$\pi_1(X, p) \cong \pi_1(A, p) \underset{\pi_1(A \cap B, p)}{*} \pi_1(B, p)$$

eg

$$\pi_1\left(\right.$$  $$\left.\right) = \underset{\langle a\rangle}{\mathbb{Z}} *_1 \underset{\langle b\rangle}{\mathbb{Z}} = F\langle a,b\rangle$$

and

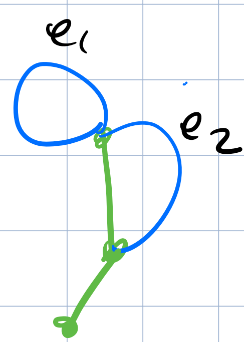
$$\pi_1\left(\right.$$  $$\left.\right) = F\langle a_1, a_2, \ldots a_n\rangle$$

Proof: induction on n.

eg X = finite graph

$\cup$

T = maximal tree



$\{e_1, \ldots, e_n\}$ = edges in $X \setminus T$

$$A_i = T \cup e_i, \quad \pi_1(A_i) = \mathbb{Z} \qquad A_i \cap A_j = T, \quad \pi_1 T = \langle 1\rangle$$

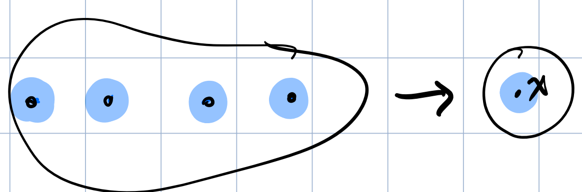
$$\underset{\text{van Kampen}}{\Rightarrow} \quad \pi_1 X \cong \mathbb{Z} * \cdots * \mathbb{Z} = F\langle e_1, \ldots, e_n\rangle$$

<u>$\pi_1(X,b)$ and covering spaces:</u>

$Y \xrightarrow{p} X$ is a <u>covering space</u> if

$\forall x \in X$, $\exists$ open nbd U_x of x

s.t. $p^{-1}(U_x) \cong \coprod U_x$.



(it is called a "space" but it's more than the space Y - it includes the map p)

eg $\quad \mathbb{R} \longrightarrow S^1$

$\quad\quad t \longmapsto e^{2\pi i t}$

If X has some basic connectedness properties, (connected, locally path connected, semi-locally simply connected), which are satisfied, eg, if $X =$ connected CW complex)

Then there is a covering $\tilde{X} \xrightarrow{p} X$ such that $\underbrace{\tilde{X} \text{ is connected and } \pi_1(\tilde{X}) = 1}$

(ie $\tilde{X}$ is is simply connected)

called the universal cover.

$\tilde{X}$ is unique up to homeomorphism, ie

If $\tilde{X}' \xrightarrow{p'} X$ is another covering space with $\tilde{X}'$ simply connected, then there is a homeomorphism $\tilde{X} \xrightarrow{h} \tilde{X}'$ with $p' \circ h = p$:

ie
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & & \tilde{X}' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$
commutes.

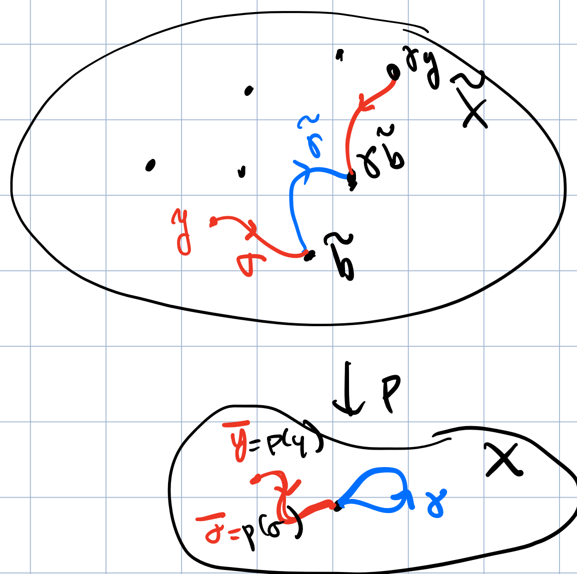
Unique path lifting:

Any covering space $Y \xrightarrow{p} X$ has unique path lifting: Given a path $\sigma : [0,1] \to X$ starting at $b \in X$ and a point $\tilde{b} \in p^{-1}(b)$, there is a unique path $\tilde{\sigma} : [0,1] \to Y$ starting at $\tilde{b}$ with $p(\tilde{\sigma}) = \sigma$.

This allows us to define an action of $\pi_1(X, b)$ on $\tilde{X}$:

Choose $\tilde{b} \in p^{-1}(b)$, and suppose $\gamma : [0,1] \to X \in \pi_1(X, b)$.

Lift γ to $\tilde{\gamma}$ starting at $\tilde{b}$.
Then $\gamma \cdot \tilde{b} = \tilde{\gamma}(1) \in p^{-1}(b)$.
If $y \neq b \in \tilde{X}$, choose a path σ from y to $\tilde{b}$ with image $\bar{\sigma} \subset X$, and lift $\bar{\sigma}.\gamma.\bar{\sigma}^{-1}$ to a path ν in $\tilde{X}$ starting at y.

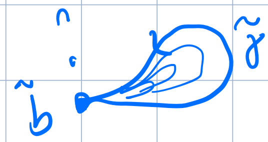


Then define $\gamma \cdot y = (\widetilde{\bar{\sigma}.\gamma.\bar{\sigma}^{-1}})(1) = (\sigma.\tilde{\gamma}.\sigma^{-1})(1)$

Claim: This is well-defined (follows because any two paths y to $\tilde{b}$ are homotopic)

Claim: The action is free and proper

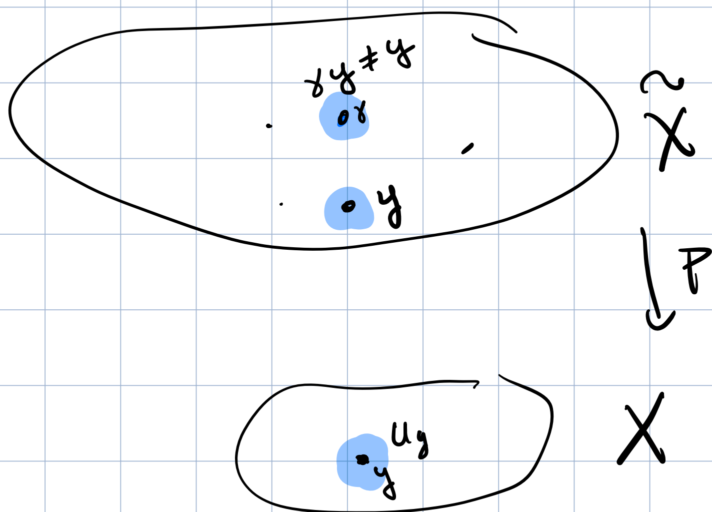
Free:



$\gamma \cdot (\tilde{b}) = \tilde{b} \Rightarrow$ can homotope $\tilde{\gamma}$ to $\tilde{b}$ $\Rightarrow$ can homotope γ to b $\Rightarrow [\gamma] = id$.

Exercise $\gamma \cdot y = y \Rightarrow \gamma \simeq b$ even if $y \neq \tilde{b}$

Proper: If $y \in \tilde{Y}$, let U_y = nbd of y evenly covered by p



Action free, so

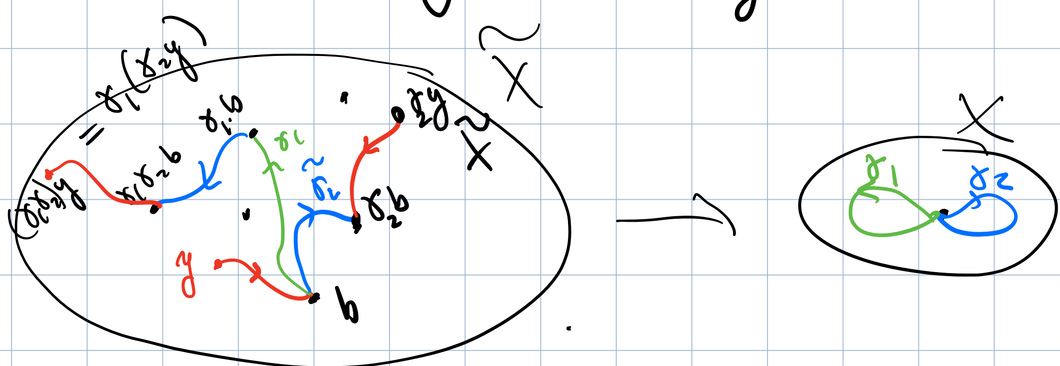
$\gamma \neq id \Rightarrow \gamma \cdot y \neq y$

$\Rightarrow \gamma U_y \cap U_y = \emptyset$.

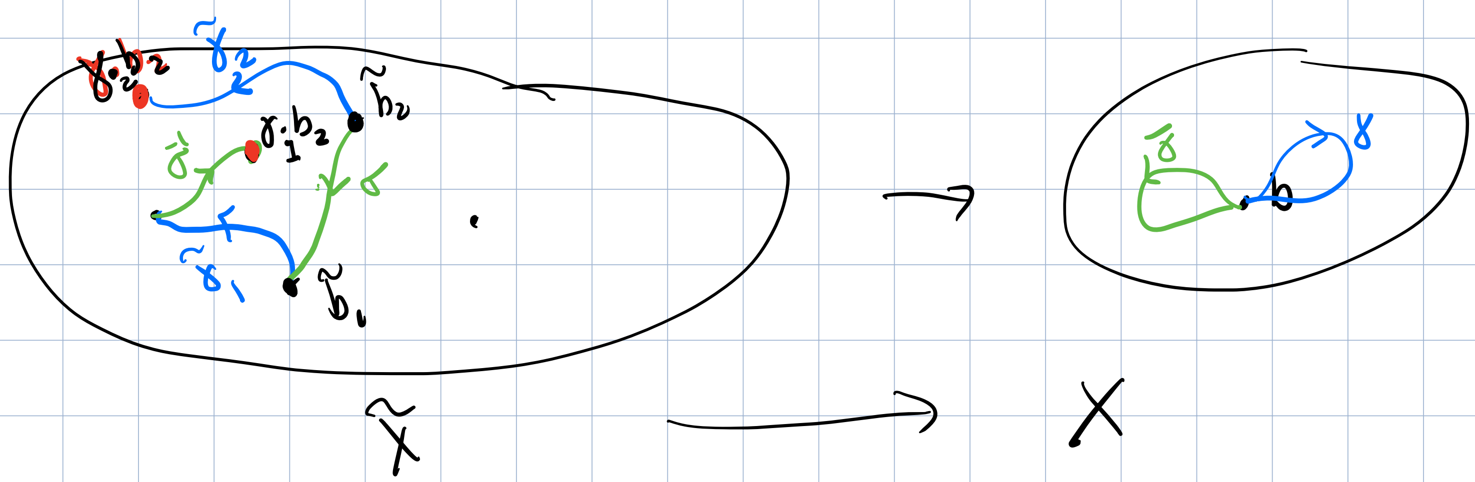
$\Rightarrow \gamma y \neq y$

Note this is a left action: $\gamma_1, \gamma_2 \in \pi_1(X, b)$

$$\Rightarrow (\gamma_1 . \gamma_2) \cdot y = \gamma_1 \cdot (\gamma_2 \cdot y)$$



(*) A <u>different</u> choice of $\tilde{b}$ gives a <u>different</u> action. (*)



(If the red points are the same, then $\tilde{\gamma}_2 \cdot \sigma \simeq \sigma \cdot \tilde{\gamma}_1$

$\Rightarrow$ $\gamma\sigma \simeq \sigma\sigma$ in X, but in general $\pi_1 X$ is not commutative (eg $\pi_1(\infty) \cong F_2$))

Finally, The action is by homeomorphisms:

$$g : \tilde{X} \longrightarrow \tilde{X} \quad \text{is continuous, and}$$
$$y \longmapsto g \cdot y$$

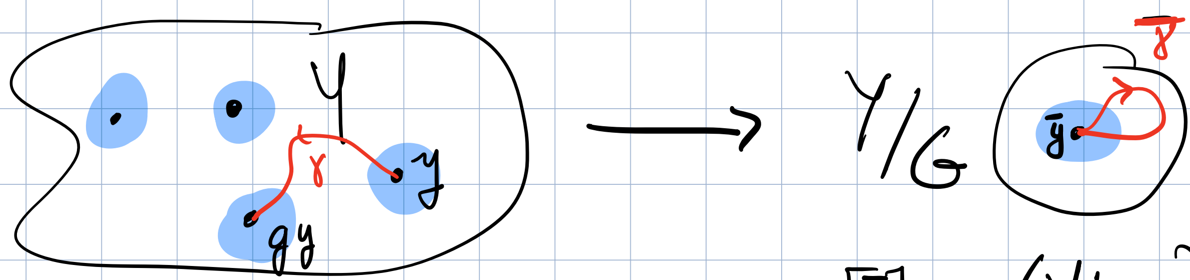
$$g^{-1} : \tilde{X} \longrightarrow \tilde{X} \quad \text{is a continuous inverse.}$$

The action of G on $\tilde{X}$ has quotient $G \backslash \tilde{X} \approx X$. In particular $\pi_1(G \backslash \tilde{X}) \cong G$.

If G acts freely and properly on a nice enough space Y, there is a sort of converse:

Proposition: Suppose Y is locally compact, Hausdorff, simply connected and locally path-connected. If G acts freely and properly on Y, then $\pi_1(G\backslash Y) \cong G$.

Proof pick $y \in Y$ and γ a path from y to gy. let $\bar{\gamma}$ be the image of γ in $G\backslash Y$ and define
$$f(g) = [\bar{\gamma}] \in \pi_1(Y/G):$$



$[\bar{\gamma}] \in \pi_1(Y/G, \bar{y})$

Any other path γ' from y to gy is homotopic to γ, so $\bar{\gamma} \simeq \bar{\gamma}'$ and $f: G \to \pi_1(G\backslash Y, \bar{y})$ is well-defined.

Claim: f is an isomorphism

Proof: First show $Y \to Y/G$ is a covering space

Since G acts freely on Y, $gy \neq y$ for all $g \in G$, $y \in Y$.
Since Y is locally compact, each $y \in Y$ has a neighborhood U_y with compact closure $\overline{U_y}$.
Since the action is proper, only finitely many translates of $\overline{U_y}$ intersect $\overline{U_y}$, so only finitely many translates of U_y intersect U_y.
Since Y is Hausdorff and the translates of y are distinct, these can be separated by smaller neighborhoods $gU_y' \subset gU_y$. The intersection of these neighborhoods is an open set V_y such that

$$gV_y \cap V_y = \emptyset \text{ for all } g \in G.$$

Therefore the image of V_y in $G \backslash Y$ is evenly covered by the translates of V_y. ✓

Since Y is locally path-connected, and simply connected, Y is the universal cover of $G \backslash Y$, and G can be identified with the group of deck transformations,

so $\pi_1(G \backslash Y) \cong G$.

Galois Correspondence

$\pi_1(X,b)$ acts freely & properly on $\tilde{X}$

so $H \leq \pi_1(X,b)$ also acts freely & properly

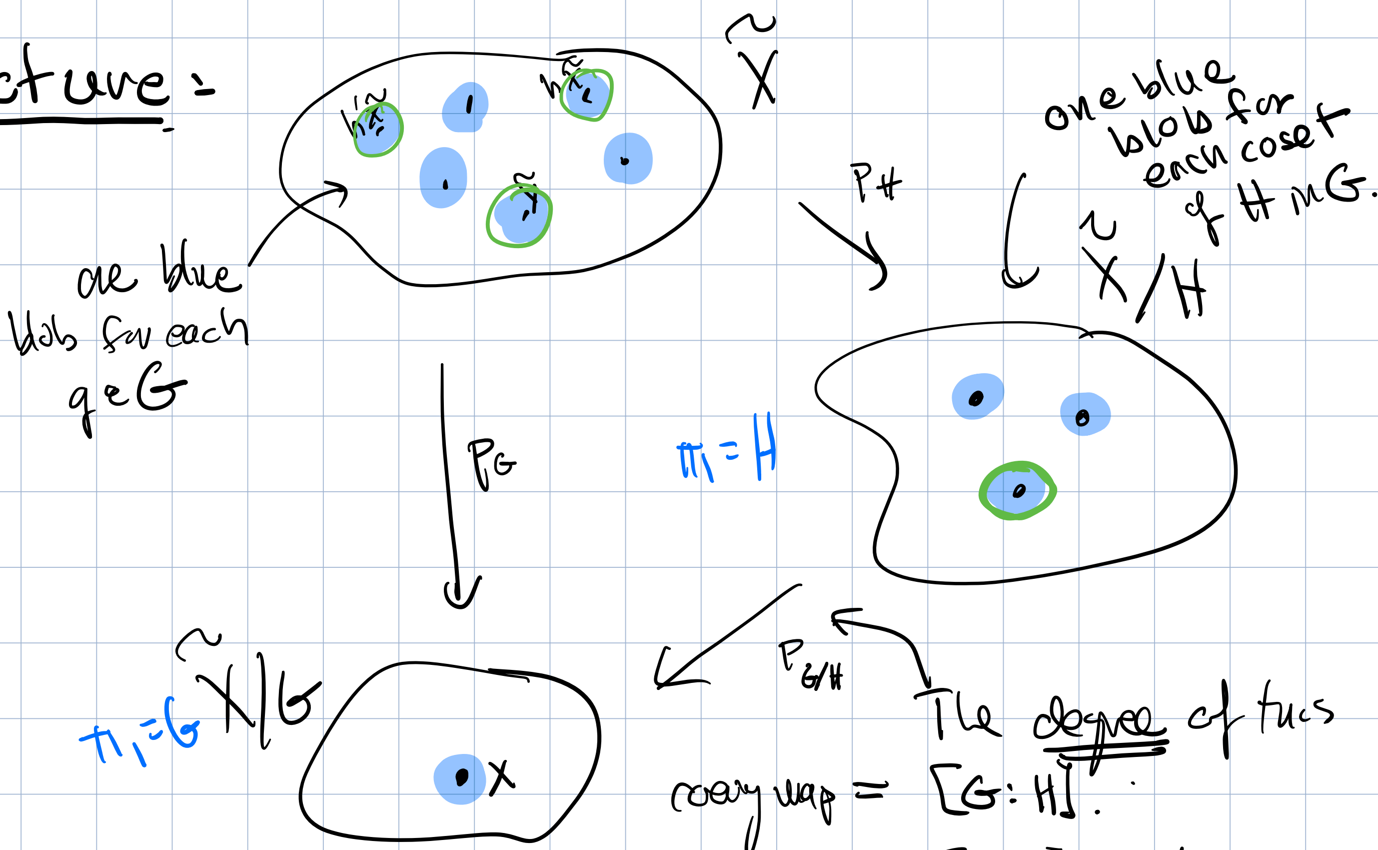
so

$$\tilde{X} \longrightarrow \tilde{X}/H \quad \text{and} \quad \tilde{X}/H \longrightarrow \tilde{X}/G = X$$

are covering spaces

and $\pi_1(\tilde{X}/H) \cong H$.

picture:



The degree of this covering map $= [G:H]$.

ie it is $[G:H]$-to-1.

Conversely: if $Y \xrightarrow{p} X$ is a covering space then the induced map $p_* \pi_1(Y,y) \longrightarrow \pi_1(Y, p(y))$ is injective, so identifies

$\pi_1(Y,y)$ with a subgroup of $\pi_1(Y, p(y))$

If $p(y') = p(y)$ for $y' \neq y$; then

$p_* \pi_1(Y, y')$ is conjugate to

$p_* \pi_1(Y,y)$.

Theorem (Galois's correspondence)

$$\left\{\begin{array}{l}\text{conjugacy classes}\\ \text{of subgroups of } \pi_1(X,b)\end{array}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{array}{l}\text{path-connected}\\ \text{covering spaces } Y \xrightarrow{p} X\end{array}\right\}$$

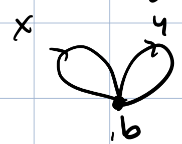
$$H \leq \pi_1 X \longrightarrow \left(\tilde{X}/H \xrightarrow{p} \tilde{X}/G\right)$$

$$p_* \pi_1 Y \leq \pi_1 X \longleftarrow \left(Y \xrightarrow{p} X\right)$$

$$\begin{tikzcd}
p^{-1}(x) \leftrightarrow \text{elts of } G \quad \tilde{X} \arrow[r, "p_1"] \arrow[dr, "p"'] & \tilde{X}/H \arrow[d, "p_2"] & \text{points in } p_2^{-1}(x) \leftrightarrow \text{cosets of } H \text{ in } G \arrow[d] \\
& \tilde{X}/G = X & \ni x
\end{tikzcd}$$

so $\# p_2^{-1}(x) = [G:H]$

Example: $X = R_2$ $\qquad \pi_1 X \cong F_2 = F\langle x, y\rangle$



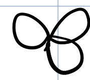
$\tilde{X}$: we just need to find a 1-connected covering space of X

Lemma: $Y \xrightarrow{p} X$ any cover $\Rightarrow$ X is a graph

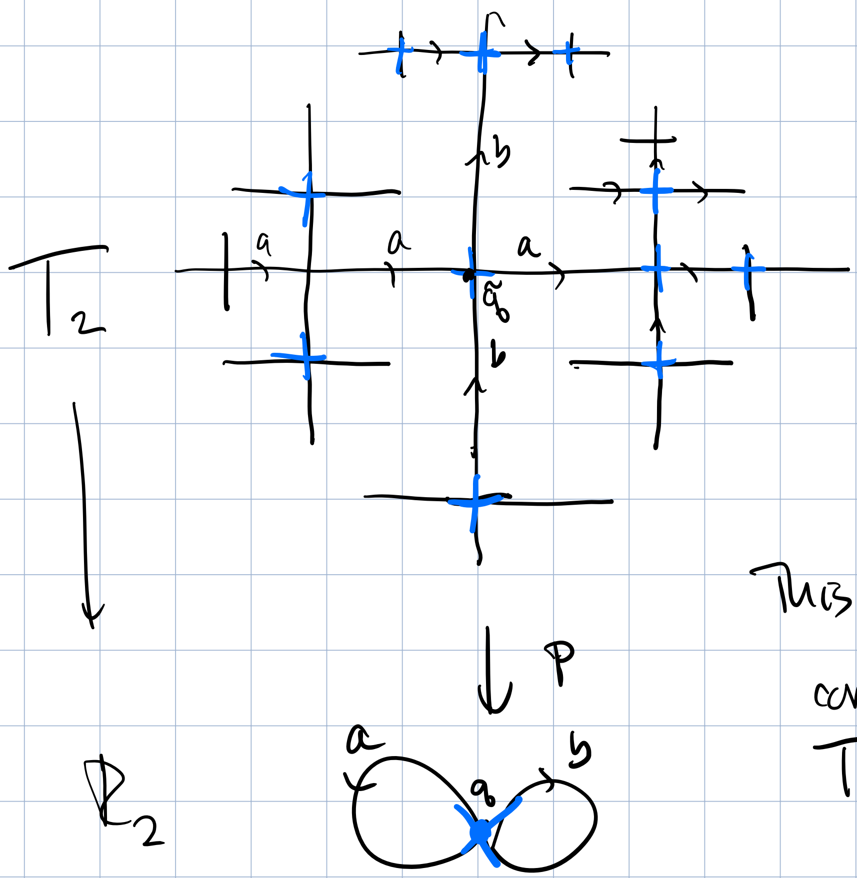
vertices: $v(Y) = p^{-1} b$

edges: $e(Y) = \{$lifts of the loops x and $y\}$.

$v(Y) \cup e(Y) \subset Y$

$p: Y \longrightarrow$  a local homeomorphism

$\pi_1 \tilde{X} = \langle 1 \rangle \Rightarrow \tilde{X}$ is a tree.
Just have to find a tree T that covers R_2, then by uniqueness we know $T \cong \tilde{X}$

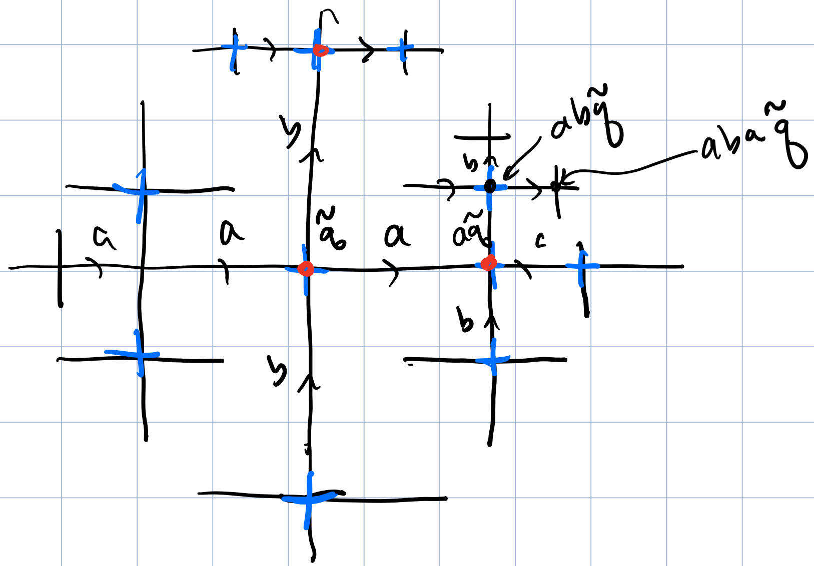


such (open) horizontal edges homeomorphically to a, vertical to b

This is a covering map, T is connected and $\pi_1 T = \langle 1 \rangle$, so T is the universal cover.

action of $\pi_1(R_2, b)$ on T:

- pick $\tilde{q} \in p^{-1}(q)$.



a shifts the whole picture to the right

b shifts the whole picture up

Using the Galois correspondence, we can prove

Theorem: Any subgroup of a free group $F(S)$ is a free group

Proof Let R_S be a wedge of circles, one for each $s \in S$, so $\pi_1(R_S) \cong F(S)$.

The universal cover $\tilde{R}_S$ is a simply-connected graph, so is a tree T_S. The action of $F(S)$ on T_S is free and cellular, so the action of any subgroup $H < F(S)$ is free and cellular, the map $T \to T/H$ is a covering map, and $H \cong \pi_1(T/H)$. But T/H is a graph, so $\pi_1(T/H)$ is free ✓