$\boxed{\text{Ping-Pong}}$

We've seen how a free proper action of G on a nice enough space Y can be used to give information about G.

But there are many actions on nice spaces that are not free. Can we still get information about G?

Here's one way:

Given an action of G on a space X (of any sort), and two elements $a, b \in G$, there is a criterion called the Ping-Pong lemma which will prove that the subgroup $\langle a,b \rangle$ of G generated by a and b is free.

This is often used to prove that G contains a (non-abelian) free group.

It suffices to prove that reduced words in a and b are never the identity in G, since then the homomorphism $F(a,b) \longrightarrow \langle a,b \rangle$ extending $\{a,b\} \longrightarrow \langle a,b \rangle$ is both surjective <u>and</u> injective.

$\boxed{\text{Ping-Pong Lemma}}$: Let $G \curvearrowright X$ and $a, b \in G$.
Suppose X contains subsets A, B, $A \not\subset B$ such that

$$\left.\begin{array}{r} a^n B \subset A \\ \text{and } b^n A \subset B \end{array}\right\} \text{ for all } n \in \mathbb{Z} \setminus \{0\}$$

Then the subgroup generated by a and b is a free group.

<u>Proof</u>: Suppose $w = a^{n_1} b^{n_2} a^{n_3} \cdots b^{n_{k-1}} a^{n_k} = id_G$, with all $n_i \in \mathbb{Z} \setminus \{0\}$ (ie w is a reduced word starting and ending with a power of a.)

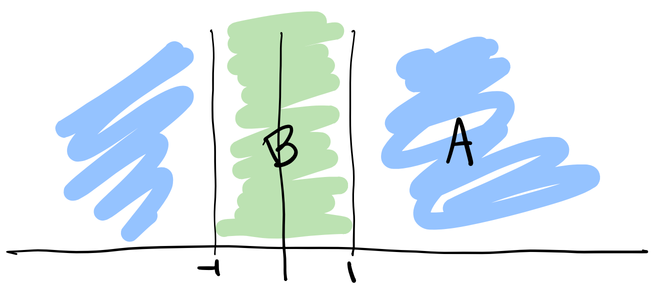
Let $x \in B \setminus A$. Then $w \cdot x \subset A$, so $wx \neq x$, so $w \neq id_G$.

If w starts or ends with b, then for large enough N, $a^N w a^{-N}$ starts <u>and</u> ends with a, so $a^N w a^{-N} \neq id_G$, so $w \neq id_G$. ∎

<u>Example</u> $G = SL(2, \mathbb{Z})$, $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

We already used the action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2$ to see reduced words are not the identity.
Let's use the action on $\mathbb{H}$ and the PP lemma

$$a \cdot z = \frac{z+2}{1}$$
$$b \cdot z = \frac{z}{2z+1}$$



$B = \{ z = r + is \mid s > 0, |r| < 1 \}$
$A = \{ z = r + is \mid s > 0, |r| > 1 \}$

Clearly $\quad a^n B \subset A$ ✓

Also need $\quad b^n A \subset B$, ie $z = r+is$, $|r|>1$, $s>0$

$$\Rightarrow \operatorname{Im}\left(\frac{z}{2z+1}\right) > 0 \text{ and } \left|\operatorname{Re}\left(\frac{z}{2z+1}\right)\right| < 1$$

Well, $\frac{z}{2z+1} = \frac{2\|z\|+z}{\|2z+1\|^2}$ has imaginary

part $\frac{\operatorname{Im}(z)}{\|2z+1\|^2}$. This is >0 since $\operatorname{Im}(z)>0$.

The real part is $\frac{\operatorname{Re}(z)+2\|z\|^2}{\|2z+1\|^2} = \frac{r+2(r^2+s^2)}{4r+1+4(r^2+s^2)}$

Need to check $|\operatorname{Re}(z)+2\|z\|^2| < \|2z+1\|^2$

ie $-4r-1-4(r^2+s^2) < r+2(r^2+s^2) < 4r+1+4(r^2+s^2)$

ie $-5r < 1+6(r^2+s^2)$ and $-3r < 1+2(r^2+s^2)$

Since $|r|>1$, $r^2+s^2 > r^2$ so it suffices to show

$-5r < 1+6r^2$ and $-3r < 1+2r^2$

This is clear if $r>1$

If $r<-1$, we just need to check that the roots of $2r^2+3r+1$ and $6r^2+5r+1$ are all ≥ -1. ✓

If the action of G on a space X is free, the orbit of any point $x \in G$ can be thought of as a copy of G

If the action is proper but not free, then point stabilizers are finite, and an orbit is "almost" a copy of G.

$\boxed{\text{Quasi-isometry}}$

Suppose G is finitely generated by S.
Recall that word length makes G into a metric space:

$$d_S(g,h) = \ell_S(g^{-1}h) = \text{length of shortest word in } S \cup S^{-1} \text{ representing } g^{-1}h$$

Thus
$$\left.\begin{aligned} & d_S(g,g) = 0 \\ & d_S(g,h) = d_S(h,g) \\ & d_S(g,h) > 0 \quad \text{if } g \neq h \\ & d_S(g,h) + d_S(h,k) \leq d_S(g,k) \end{aligned}\right\} \Rightarrow d_S \text{ is a metric}$$

G acts on (G, d_S) by left multiplication, which is an isometry: $d_S(g\cdot x, g\cdot y) = \ell_S(y^{-1}g^{-1}gx) = \ell_S(y^{-1}x) = d_S(x,y)$

If X is a metric space and $G \curvearrowright X$ then we can measure the distance in X between points in an orbit Gx.

We know
$$\begin{aligned} (G, d_S) &\longrightarrow (Gx, d_X) \\ g &\longmapsto gx \end{aligned}$$
is a surjective map. It is probably not an isometry but if X is "nice" and $G \curvearrowright X$ is proper and cocompact we will show it is "close" to an isometry, specifically it is a quasi-isometry

Def X, Y metric spaces, $f: X \to Y$ is a quasi-isometric embedding if there are constants $\lambda \geq 1$, $c \geq 0$ such that:

$$\boxed{\frac{1}{\lambda} d(x,x') - c \leq d(f(x), f(x')) \leq \lambda d(x,x') + c}$$

f is quasi-surjective if there is a constant K s.t. $\forall y \in Y$ $\exists x \in X$ with $d(y, f(x)) \leq K$.

A quasi-isometric embedding that is quasi-surjective is a quasi-isometry.

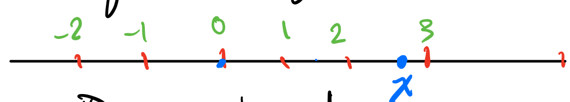
Also called a (λ, c)-quasi-isometry. Note a (λ, c)-qi is a (λ', c')-qi for any $\lambda' \geq \lambda$, $c' \geq c$.

eg $\mathbb{Z}$ generated by $S = \{t\}$ with the word metric

$$d_S(t^m, t^n) = |m - n|$$

$f: \mathbb{Z} \longrightarrow \mathbb{R}$, $t^n \longmapsto n$ the natural embedding preserves distance, ie is a quasi-isomorphic embedding with $\lambda = 1$, $c = 0$.

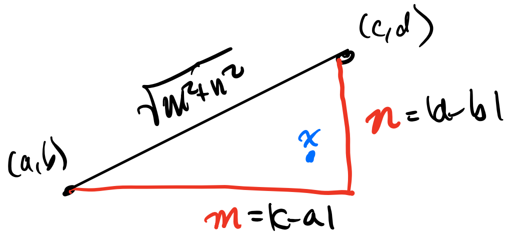
f is also quasi-surjective: take $K = 1/2$



every $x \in \mathbb{R}$ is within $\frac{1}{2}$ of the image $f(\mathbb{Z})$.

eg $\mathbb{Z}^2$, $S=\{s,t\}$, $d_S(s^a t^b, s^c t^d) = |c-a|+|b-d|$

$$(\mathbb{Z}^2, d_S) \longrightarrow (\mathbb{R}^2, d_S)$$
$$s^a t^b \longmapsto (a,b)$$



$(m, n \geq 0)$

$K=\frac{\sqrt{2}}{2}$, $\lambda=\sqrt{2}$, $C=0$ works.

$$d_{\mathbb{R}}((a,b),(c,d)) = \sqrt{m^2+n^2} \leq m+n \quad ✓$$
$$\geq \frac{1}{\sqrt{2}}(m+n) \quad ✓$$ (square both sides and simplify)

$$\left(\begin{array}{l} 2(m^2+n^2) \geq m^2+2mn+n^2 \\ m^2+n^2-2mn = (m-n)^2 \geq 0 \end{array} \right)$$

<u>Proposition</u> Quasi-isometry is an equivalence relation on metric spaces.

<u>Proof</u>

① Reflexive (use id_X) ✓

② Transitive, ie

<u>Claim</u> if $f: X \to Y$, $g: Y \to Z$ are quasi-isometries then $g \circ f: X \to Z$ is a quasi-isometry

<u>proof</u>

We need to find λ, C st.

$$\frac{1}{\lambda} d_X(x,x') - C \leq d_Z(gf(x), gf(x')) \leq \lambda\, d_X(x,x') + C$$

RH inequality:

$$d_Z(gf(x), gf(x')) \leq \lambda_g d_Y(f(x), f(x')) + C_g$$
$$\leq \lambda_g(\lambda_f d_X(x,x') + C_f) + C_g$$
$$= \underbrace{\lambda_g\lambda_f}_{\lambda_1} d_X(x,x') + \underbrace{\lambda_g C_f + C_g}_{C_1}$$

LH inequality

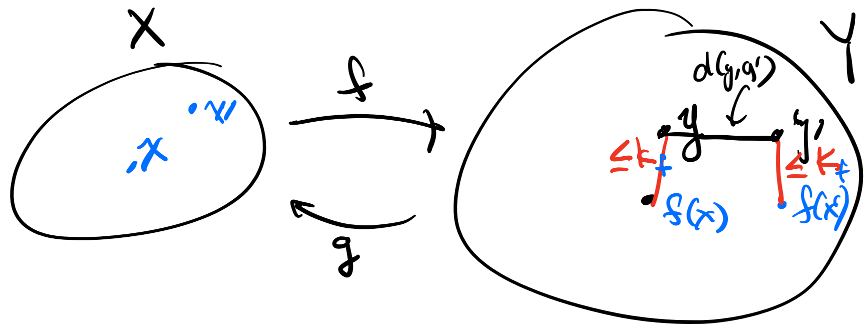
$$\frac{1}{\lambda} d_X(x,x') - C \leq d_Z(gf(x), gf(x'))$$
$$\Leftrightarrow \quad d_X(x,x') \leq$$

$$d_Z(gf(x), gf(x')) \geq \frac{1}{\lambda_g} d_Y(f(x), f(x')) - C_g$$
$$\geq \frac{1}{\lambda_g}\left(\frac{1}{\lambda_f} d_X(x,x') - C_f\right) - C_g$$
$$= \underbrace{\frac{1}{\lambda_g\lambda_f}}_{\lambda_2}(d_X(x,x')) - \underbrace{\left(\frac{C_f}{\lambda_g} + C_g\right)}_{C_2'}$$

Now take $\lambda = \max(\lambda_1, \lambda_2)$
$C = \max(C_1, C_2)$ ✓

③ Symmetric, ie
<u>Claim</u>: $f: X \to Y$ a quasi-isometry
Then $\exists\, g: Y \to X$ a quasi-isometry

Proof



For each $y \in Y$ choose x with $d(f(x),y) \leq K_f$, and define $g(y) = x$

we claim that g is a quasi-isometry

① find λ_1, C_1 with

$$d(\underset{x}{g(y)}, \underset{x'}{g(y')}) \leq \lambda\, d(y,y') + C_1$$

we know

$$\frac{1}{\lambda_f} d(x,x') - C_f \leq d(f(x), f(x'))$$

$$d(x,x') \leq \lambda_f \big(d(f(x),f(x')) + C_f\big)$$

$$\leq \lambda_f \big(d(y,y') + 2K_f + C_f\big)$$

$$\therefore \quad d(x,x') \leq \underbrace{\lambda_f}_{\lambda_1} d(y,y') + \underbrace{\lambda_f(2K_f + C_f)}_{C_1}$$

② find λ_2, C_2 with

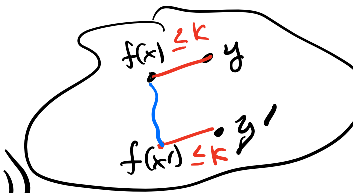
$$\frac{1}{\lambda} d(y,y') - C \leq d(\underset{x}{g(y)}, \underset{x'}{g(y')})$$

We know $d(f(x), f(x')) \leq \lambda_f d(x,x') + C_f$

so $\frac{1}{\lambda_f}\big(d(f(x),f(x')) - C_f\big) \leq d(x,x')$

and

$$\frac{1}{\lambda_f}\left(d(y,y')-2K_f-C_f\right) \le \frac{1}{\lambda_f}\left(d(f(x),f(x'))-C_f\right)$$

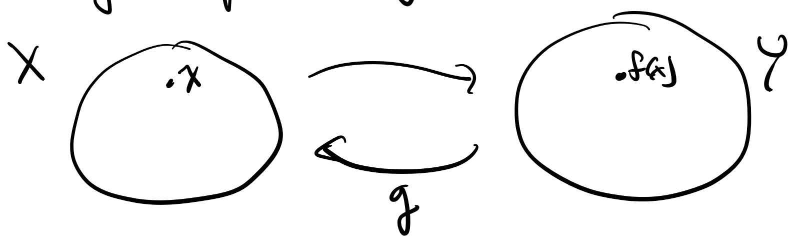


$$\therefore \quad \underset{\lambda_2}{\frac{1}{\lambda_f}}\, d(y,y') - \underset{C_2}{\frac{2K_f+C_f}{\lambda_f}} \le d(g(y),g(y'))$$

so take $\lambda=\lambda_f$, $C=\max\left(\frac{2K_f+C_f}{\lambda_f}, \lambda_f(2K_f+C_f)\right)$

$= \lambda_f(2K_f+C_f)$

③ Show g is quasi-surjective, with some constant K

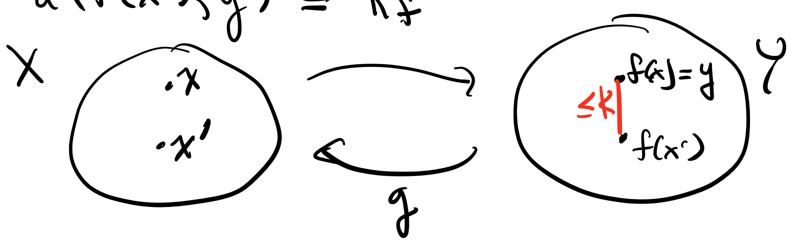


$x\in X$. Need to find $y \in Y$ with $d(g(y),x)\le K$

The obvious thing to try is $y=f(x)$.

We defined $g(y)$ by picking some x' with

$d(f(x'),y) \le K_f$



Then $\quad \frac{1}{\lambda_f} d(x,x') - C_f \le d(f(x),f(x')) \le K_f$

so $\quad d(x,\underset{g(y)}{\underset{\shortparallel}{x'}}) \le \underbrace{\lambda_f(K_f+C_f)}_{K}$

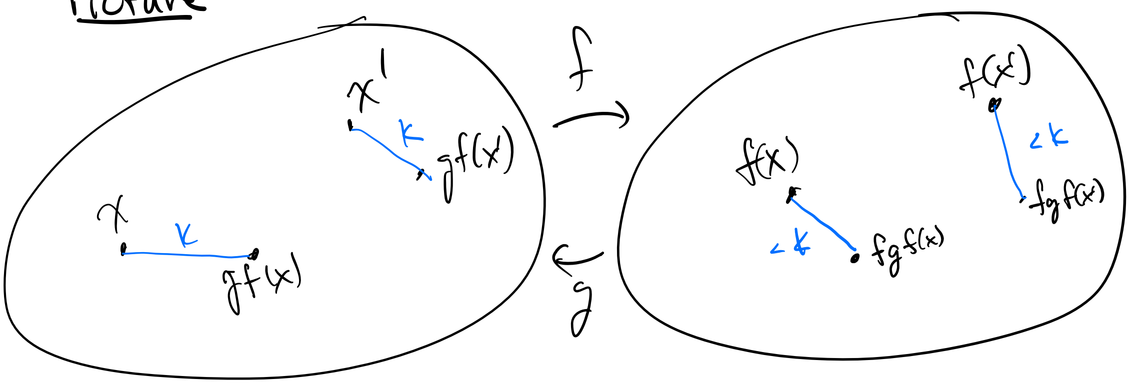
$d(x,g(y)) \le K$ ✓

Prop X, Y metric spaces, $f: X \to Y$, $g: Y \to X$ such that

1. $d(f(x), f(x')) \le \lambda d(x, x') + C$
2. $d(g(y), g(y')) \le \lambda d(y, y') + C$
3. $d(x, gf(x)) \le K$
4. $d(y, fg(y)) \le K$

Then g, f are quasi isometries.

<u>Picture</u>



<u>Proof</u>

Let's show f is a quasi-isometry

(the argument is symmetric for g)

We <u>have</u> λ, C st. $d_Y(f(x), f(x')) \le \lambda d_X(x, x') + C$

We <u>need</u> λ', C' st.

$$d_Y(f(x), f(x')) \geq \frac{1}{\lambda'} d_X(x, x') - C'$$

<u>ie</u> $d_X(x, x') \le \lambda' d_Y(f(x), f(x')) + \lambda' C'$

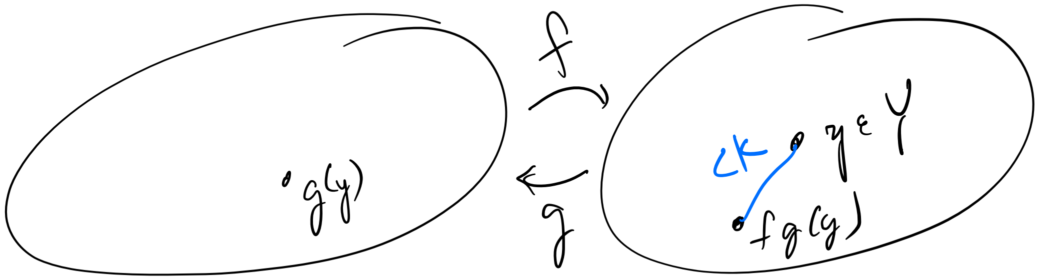
Well, $d(x,x') \leq d(x, gf(x)) + d(gf(x), gf(x')) + d(gf(x'), x')$

$$\leq d(gf(x), gf(x')) + 2K$$

$$\leq \underbrace{\lambda}_{\lambda'}\, d(f(x), f(x')) + \underbrace{C + 2K}_{C'}$$

✓

and: f is quasi-surjective:



If $y \in Y$, then $d(y, fg(y)) < K$ ✓

Application:

<u>Prop</u> S, S' finite generating sets for G

Then $f := id : (G, d_S) \longrightarrow (G, d_{S'})$ is a quasi-isometry.

<u>Pf</u>: If $g : (G, d_{S'}) \longrightarrow (G, d_S)$ is also $= id$, then $d(y, fgy) = d(x, gfx) = 0$ so f and g satisfy ③ and ④ in the proposition.

For ①, need to find λ, C s.t. for all $g, h \in G$

$$d_{S'}(g_1,g_2) \leq \lambda\, d_S(g_1,g_2) + C$$

ie $$d_{S'}(1, \underbrace{g_1^{-1}g_2}_{h}) \leq \lambda\, d_S(1, \underbrace{g_1^{-1}g_2}_{h}) + C$$

Write each $s \in S$ as a word in S':

$$s = w_{S'}(s)$$

Let $$m = \max_{s \in S} \left(\text{length of } w_{S'}(s)\right)$$

Suppose $h = s_1 \cdots s_k$ is a shortest word,

ie $d_S(1,h) = k$

Then $h = w_{S'}(s_1) \cdots w_{S'}(s_k)$ is a word in S' representing h of length $\leq k \cdot m$.

so $$d_{S'}(1,h) \leq k \cdot m = \underbrace{m}_{\lambda} \cdot d_S(1,h) + \underbrace{0}_{C}$$ ✓

Condition ② is entirely symmetric:

write $s' \in S'$ as a word in S, to get

$$d_S(1,g) \leq m'\, d_{S'}(1,g)$$ ✓

We're ready for the fundamental theorem of GGT.
First here is a vague statement:

> If X is a nice metric space
> and $G \curvearrowright X$ is a proper
> cocompact action,
> Then G is quasi-isometric to X

To get a precise statement we need to say
what "nice" means...

Let X be a metric space.

A path from x to y in X is a geodesic
if its length is equal to $d(x,y)$.

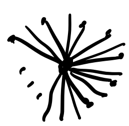
(Geodesics are not necessarily unique!)

A metric space is geodesic if any two points
are connected by a geodesic

A metric space is proper if balls B_r of
finite radius r are compact.

non-examples: • $\mathbb{R}^2 - \{(0,0)\}$ is not geodesic

• a graph with an infinite-valent vertex is not
proper

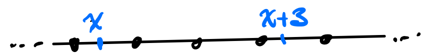


Thm (Švarc-Milnor Lemma) (X, d_X) = proper geodesic metric space, $G \curvearrowright X$ a proper cocompact action. Then G is finitely generated and (G, d_S) is quasi-isometric to (X, d_X) for any finite generating set S.

<u>examples</u> (Exercise)

$\mathbb{Z} \curvearrowright \mathbb{R}$

$n \longmapsto (x \mapsto x+n)$



Translates of $[0,1]$ cover $\mathbb{R}$, so $(\mathbb{Z}, d_S)$ is q.i. to $\mathbb{R}$

$D_\infty \curvearrowright \mathbb{R}$

$r \longmapsto (x \mapsto -x)$

$t : \quad (x \mapsto x+t)$

Translates of $[0, \frac{1}{2}]$ cover $\mathbb{R}$ so (D, d_S) is also q.i. to $\mathbb{R}$

<u>Proof of Theorem:</u> Choose any $x \in X$ and define $f : G \longrightarrow X$ by $f(g) = g \cdot x$

<u>Claim 1</u> G is finitely generated.

<u>Claim 2</u> For some (hence any) finite generating set S, $f : (G, d_S) \longrightarrow (X, d_X)$ is a quasi-isometry

Choose $K \subset X$ compact s.t. $X = \bigcup_g gK$, with $x \in K$.

Then $K \subset U$ for some open ball $U = B_r(x)$, with (compact) closure D.

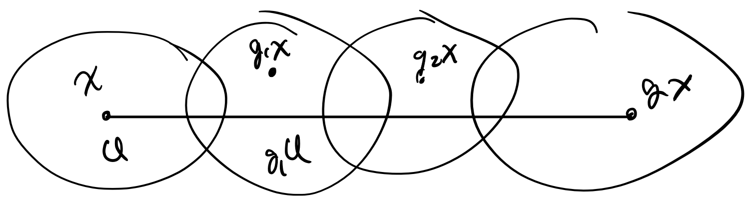
Now $X = \bigcup_{g \in G} gU$ and

$S = \{g \in G \mid gU \cap U \neq \emptyset\} \subseteq \{g \in G \mid gD \cap D \neq \emptyset\}$,

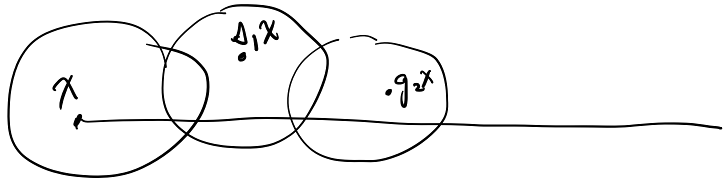
which is finite since the action is proper

<u>Claim 1</u>: S generates G.

<u>proof</u> For $g \in G$ let σ be a path from x to gx. Cover σ by translates of U, then take a finite subcover $U, g_1U, g_2U, \ldots, gU$



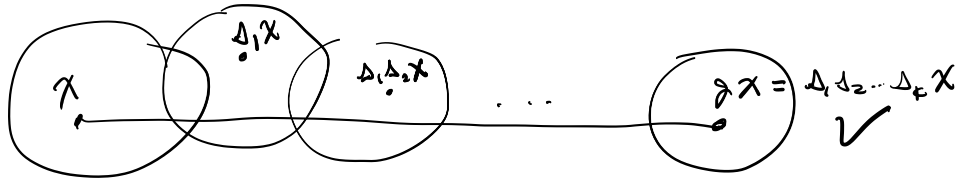
$U \cap g_1U \neq \emptyset \Rightarrow g_1 \in S$, say $g_1 = s_1$



$s_1U \cap g_2U \neq \emptyset \Leftrightarrow U \cap s_1^{-1}g_2U \neq \emptyset$

$\Rightarrow s_1^{-1}g_2 = s_2 \in S \Rightarrow g_2 = s_1s_2$

etc Continue, get $g = s_1 s_2 \cdots s_k$



Claim 2: $f : (G, d_s) \longrightarrow (X, d_X)$ is a quasi-isometry.

We need to find λ, C, K such that $\forall g_1, g_2 \in G$

① $$d_X(g_1 x, g_2 x) \leq \lambda\, d_s(g_1, g_2) + C$$

Since G acts by isometries, this is the same as

$$d_X(x, \underbrace{g_1^{-1} g_2}_{h} x) \leq \lambda\, d_s(1, \underbrace{g_1^{-1} g_2}_{h}) + C$$

② $$d_X(x, hx) \geq \frac{1}{\lambda} d_s(1, h) - C$$

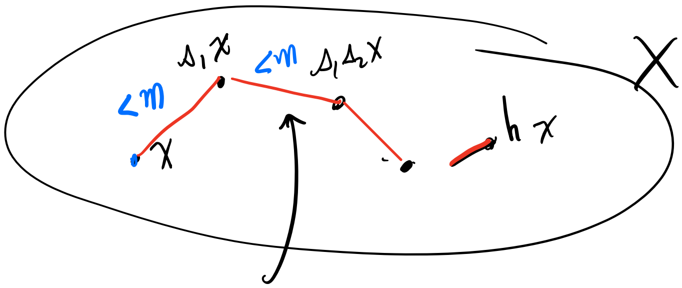
and ③ $\forall y \in X, \exists g$ s.t. $d(y, gx) < K$.

③ is easy because the balls $gB_r(x)$ cover X. ie, if $y \in X$ then there is some g with

$$d(y, gx) \leq r.$$

① Let S be (any) finite generating set. Suppose $h = s_1 \ldots s_k$ is a shortest word in S representing h, ie $d_s(1, h) = k$

Let $m = \max_{s \in S} d(x, sx)$. Then we can make a path from x to hx consisting of k segments of length $\leq m$:



$$d(s_1 x, s_1 s_2 x) = d(x, s_1 x) < m$$

etc.

so $\quad d_X(x, hx) \le m\, d_S(1,h) + 0$ ✓

② $\quad d_X(x, hx) \ge \frac{1}{\lambda} d_S(1,h) - c$

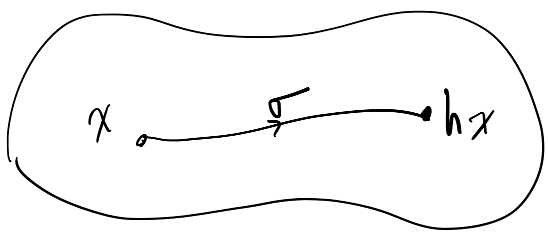
$$\iff d_S(1,h) \le \lambda\, d_X(x, hx) + \cancel{c\lambda}\, C$$

We have r s.t. $\{g \cdot B_r(x)\}$ cover X

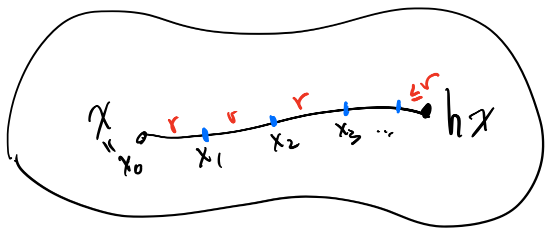
Let $S' = \{ g \mid g B_{3r}(x) \cap B_{3r}(x) \neq \emptyset \}$

This is still finite by properness of the action, and contains our old S, so still generates G.

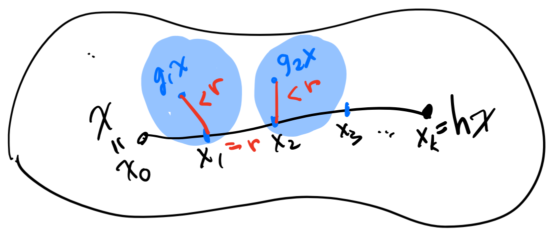
Now let σ be a geodesic from x to hx.



Divide σ into k pieces, each of length $= r$ except the last has size $r' \le r$. Now $d(x, hx) = \text{length}(\sigma) = (k-1)\cdot r + r'$



The $gB_r(x)$ cover X, so for each x_i, $\exists\, g_i$ with $d(x_i, g_i x) < r$



Then $d(g_i x, g_{i+1} x) < 3r$, so

$d(x, g_i^{-1} g_{i+1} x) < 3r$,

ie $B_{3r}(x) \cap g_i^{-1} g_i B_{3r}(x) \neq \emptyset$, so $g_i^{-1} g_{i+1} \in S'$

and $h = (1 \cdot g_1)(g_1^{-1} g_2)(g_2^{-1} g_3) \ldots (g_{k-1}^{-1} h)$

$= s_1 \quad s_2 \quad \ldots \quad s_k$

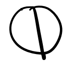
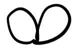
therefore $d_{S'}(1, h) \le k$

But $r(k-1) \le d_X(x, hx) \Rightarrow k \le \frac{1}{r} d_X(x, hx) + 1$

so $d_S(1, h) \le \frac{1}{r} d_X(x, hx) + 1$ ✓

Examples:

0. $\mathbb{Z} \sim \mathbb{R}$, $\mathbb{Z}^n \sim \mathbb{R}^n$

1. $F_2 = \pi_1($  $) \sim$ trivalent tree
$= \pi_1($  $) \sim$ 4-valnt tree

$\Rightarrow$ trivalnt, 4-valnt trees are q.i.

exercise: Find an explicit q.i.

$F_3 = \pi_1($  $) \sim$ trivalnt tree

$\Rightarrow F_2 \sim F_3$.

exercise: Find an explicit quasi-isometry.

Švarc-Milnor gives us a way to prove groups are quasi-isometric.

How can you prove they are <u>not</u> quasi-isometric?

eg Is $\mathbb{Z}^2 \underset{qi}{\sim} \mathbb{Z}$?

We know $\mathbb{Z}^2 \underset{qi}{\sim} \mathbb{R}^2$, $\mathbb{Z} \underset{qi}{\sim} \mathbb{R}$.

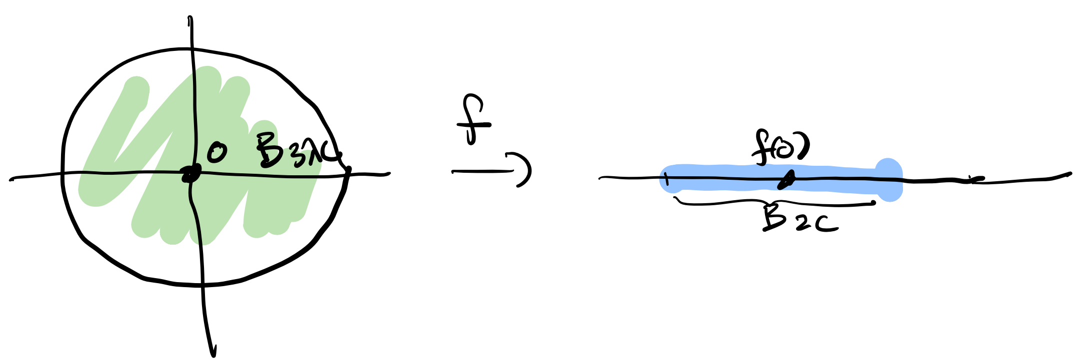
<u>Claim</u> $\mathbb{R}^2$ is <u>not</u> qi to $\mathbb{R}$

<u>proof</u> Suppose $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a quasi-isometry, with λ, C s.t.

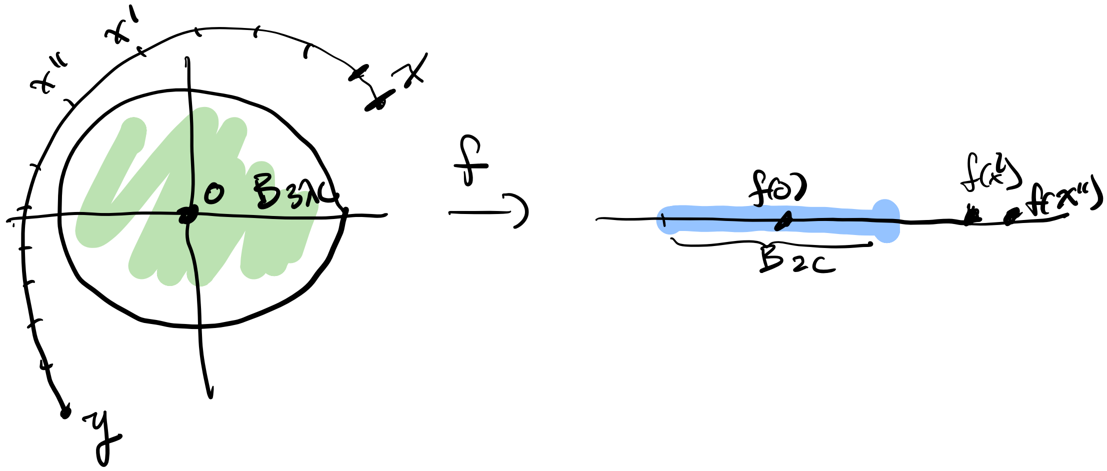
$$\frac{1}{\lambda} d(x,y) - C \leq d(f(x), f(y)) \leq \lambda d(x,y) + C$$

If $d(0,x) \geq 3\lambda C$, LHS $\Rightarrow$ $d(f(0), f(x)) \geq 2C$

If $d(x', x'') < \frac{C}{\lambda}$, RHS $\Rightarrow$ $d(f(x'), f(x'')) \leq 2C$



Let x, y lie outside $B_{3\lambda c}$ and connect them by a path outside $B_{3\lambda c}$, chopped into pieces of size $< \frac{c}{\lambda}$



Then every piece maps to the same side of B_{2c}
so $f(x), f(y)$ are on the same side of B_{2c}
$\Rightarrow$ f is not quasi-surjective. $\divideontimes$

This is a special case of the phenomenon that quasi-isometric spaces have the same number of "ends":

Cayley graphs

To use S-M, need a proper geodesic metric space. with a G-action. G acts on (G, d_S) but (G, d_S) is not geodesic!

However... Given any G with generating set S, we can define the ==Cayley graph $\mathcal{C}(G,S)$== by:

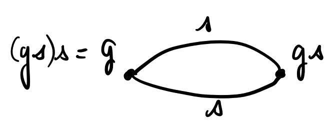
1. vertices = elements of G
2. edges: There is an edge joining g and h whenever $h = gs$ for some $s \in S$

Standard assumptions:

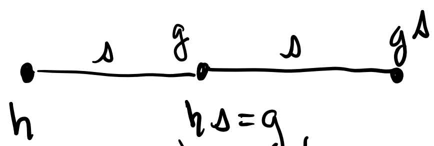
(*) $e_G \notin S$ – so there are no loops in $\mathcal{C}(G,S)$.

(*) If $s \in S$ then $s^{-1} \notin S$ (unless $s^{-1} = s$!)

If $s^2 = e_G$ get $\underline{\underline{2}}$ edges



Note: every vertex has two adjacent edges labeled s, for each $s \in S$,



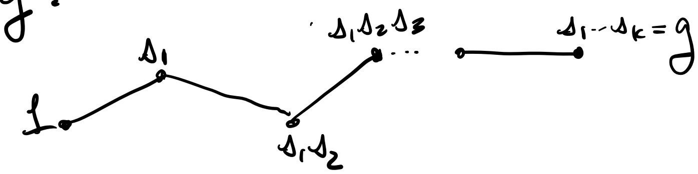
$\underbrace{hs=g}$ has a unique solution $h = gs^{-1}$.

Make $\mathcal{C}(G,S)$ into a metric space by making every edge isometric to $[0,1] \subset \mathbb{R}$; this allows you to compute the length of a path, then define $d_{\mathcal{C}}(x,y)$ = length of shortest path x to y.

G acts freely on the left on $\mathcal{C}(G,S)$:

$$g' \cdot (g \longrightarrow gs) = g'g \longrightarrow g'gs$$

For $g \in G$, write $g = s_1 s_2 s_3 \cdots s_k$ for $s_i \in S \cup S^{-1}$. Then there is a path in $\mathcal{C}(G,S)$ of length k from 1 to g:



so S generates $\Rightarrow$ $\mathcal{C}(G,S)$ is connected

Conversely, if $d_{\mathcal{C}(G,S)}(1,g) = k$, then $g = s_1 \cdots s_k$.

so $d_S(g,h) = d_S(1, g^{-1}h) = d_{\mathcal{C}(G,S)}(1, g^{-1}h)$
$= d_{\mathcal{C}(G,S)}(g,h)$

And the action of G on $\mathcal{C}(G,S)$ is by isometries.

Lemma: the inclusion $G \hookrightarrow \mathcal{C}(G,S)$ is a quasi-isometry.

Proof: it is an isometric embedding, and quasi-surjective. ✓

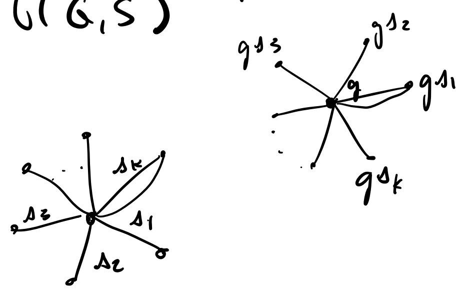
Z There is also a directed version of Cayley graphs, where you put an arrow

$$g \longrightarrow g\cdot s$$

People using this version usually assume $S = S^{-1}$. Then there are 2 edges between each pair of adjacent vertices: $hs^{-1} = g \rightleftarrows gs = h$

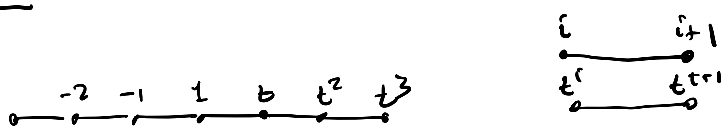
The action of G on $\mathcal{C}(G,S)$ is free and cellular, so proper (exercise)

If is also <u>cocompact</u> - translates of the closed ball of radius 1 around e_G cover $\mathcal{C}(G,S)$



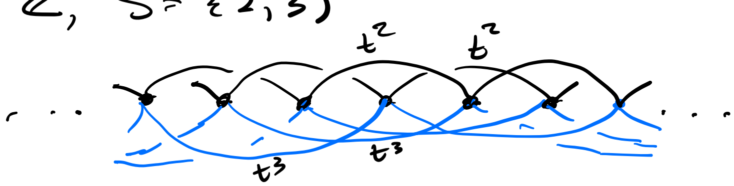
So if S is finite we have a proper cocompact action on a geodesic metric space, and we could also apply Svarc-Milnor to conclude $\mathcal{C}(G,S)$ is q.i. to (G,d_S).

<u>Examples:</u> $G = \mathbb{Z} = \langle t \rangle$ $\quad S = \{1\} = \{t\}$

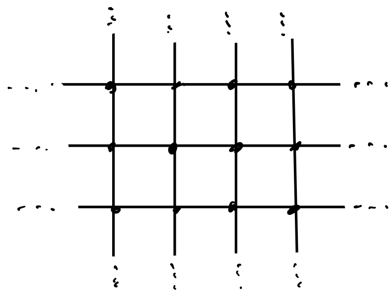


Different generating sets often give different Cayley graphs

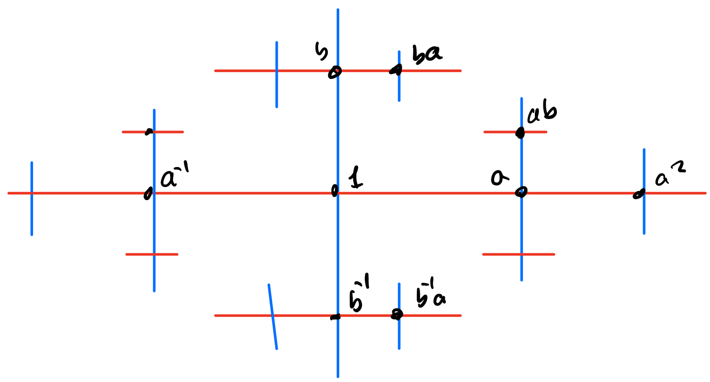
$G = \mathbb{Z}$, $S = \{2,3\}$



$G = \mathbb{Z}^2$, $S = \{(1,0), (0,1)\}$

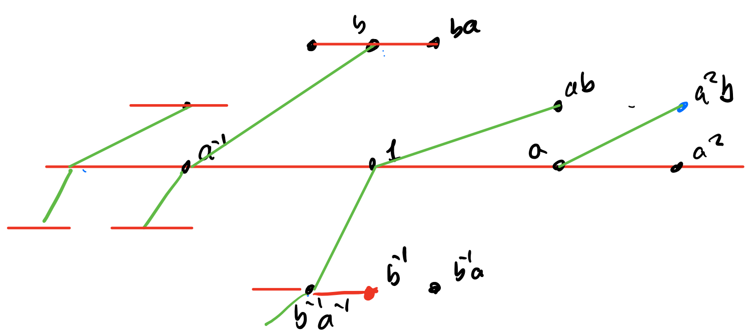


$G = F\langle a, b\rangle$, $S = \{a, b\}$

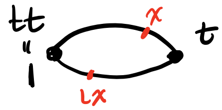


$G = F\langle a, b\rangle$, $S = \{a, ab\}$

$b^{-1} = b^{-1}a^{-1}a$

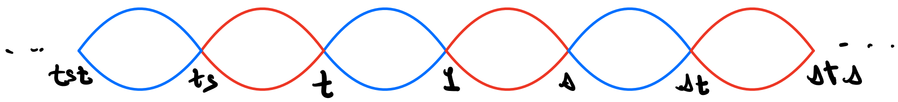


$G = \mathbb{Z}/2\mathbb{Z} = \{1, t\}$ $\qquad S = \{t\}$



$G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ $\qquad S = \{t, s\}$

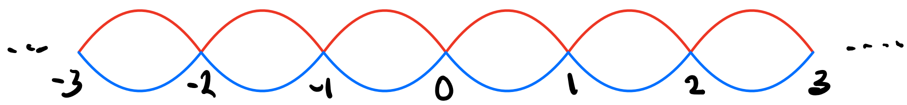
$\{1, t\}$ $\quad \{1, s\}$



Different groups can have isometric Cayley graphs:

Examples:

① $\mathbb{Z}$, $S = \{1, -1\}$



② G, G' finite, no elts of order 2, $|G| = |G'|$.
For S take all elements of G, then throw out one of each pair $\{s, s^{-1}\}$. Then $\mathcal{C}(G, S)$ is the complete graph on $|G|$ elements.
Do the same thing for G'. Then $\mathcal{C}(G, S) = \mathcal{C}(G', S')$

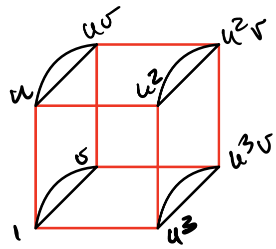
An example using minimal generating sets:

$\mathbb{Z}_4 \times \mathbb{Z}_2$

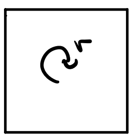
$\langle u \rangle \quad \langle v \rangle$

$S = \{u, v\}$

$\mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_2, S) =$



D_8 = symmetries of a square

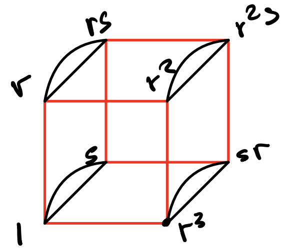


$r^4 = 1$

$s^2 = 1$

$S = \{r, s\}$

$\mathcal{C}(D_8, S) =$



Application to quasi-isometry:

G finitely generated by S, $\mathcal{C} = \mathcal{C}(G,S)$

Suppose $H < G$ has finite index, and let $\{g_0 = e_G, g_1, \dots, g_N\}$ be a set of coset representatives.

H also acts properly (free and cellular), and H-translates of $K = \bigcup_i g_i \cdot \overline{B_1(e_g)}$

cover $\mathcal{C}(G,S)$, so the action is cocompact.

Therefore, by the Svarc-Milnor Lemma,

$$H \underset{qi}{\sim} \mathcal{C}(G,S) \sim G$$

So we've proved

<u>Proposition</u> Let G be a finitely-generated group and $H \leq G$ a finite-index subgroup. Then H is quasi-isometric to G.

Ends of a metric space

Definition Let X be a proper geodesic metric space and $x_0 \in X$.

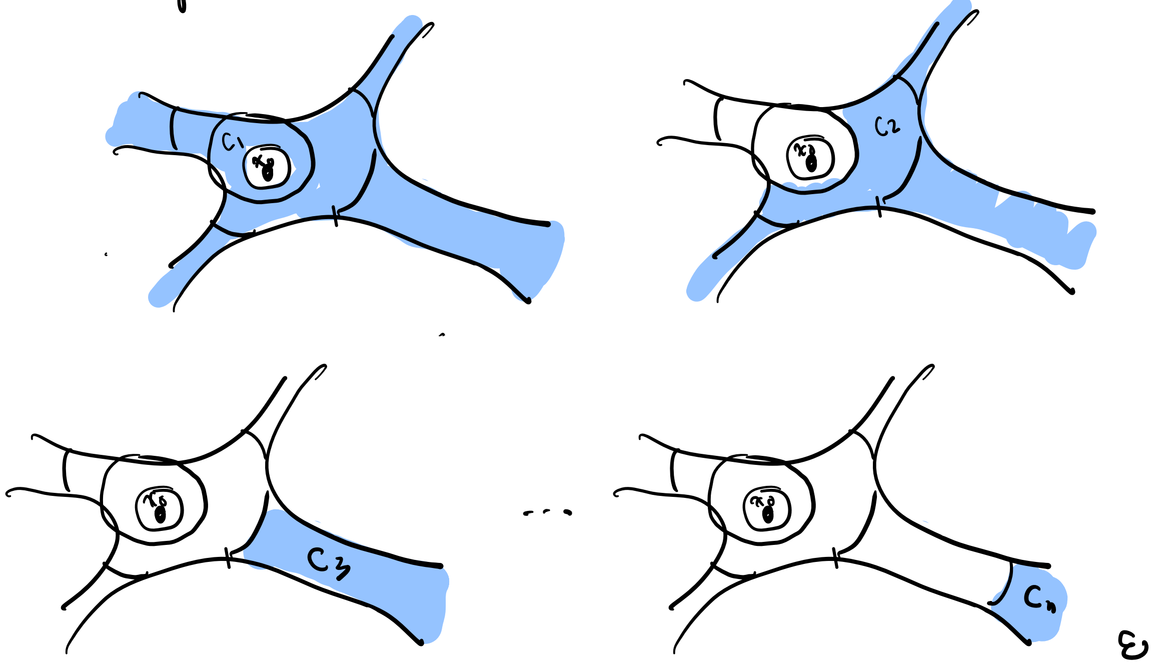
Let $B_n = B_n(x_0) =$ ball of radius n.

Each $\overline{B}_n$ is compact and $\bigcup_{n>0} B_n = X$.

An end of X is an infinite sequence

$$\varepsilon = C_1 \supseteq C_2 \supseteq \cdots$$

where each C_n is a non-$\emptyset$ connected component of $X \setminus \overline{B}_n$



note that C_n determines $C_1, C_2, \ldots, C_{n-1}$

If $C_1' \supset C_2' \supset \dots$ is a sequence of components of $X \setminus B_i(x_0')$ for some $x_0' \neq x_0$,

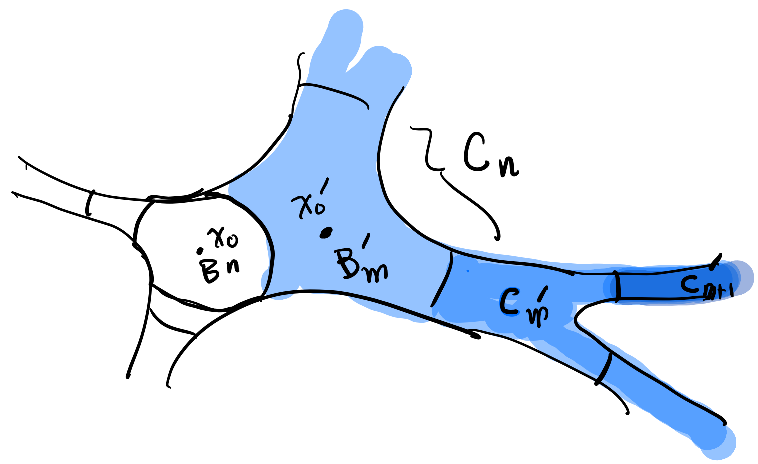
then it determines a unique end $C_1 \supset C_2 \supset \dots$, as follows:

For each n, $\exists\, r_n$ s.t. $m \geq r_n \Rightarrow B_n(x_0) \subseteq B_m(x_0')$

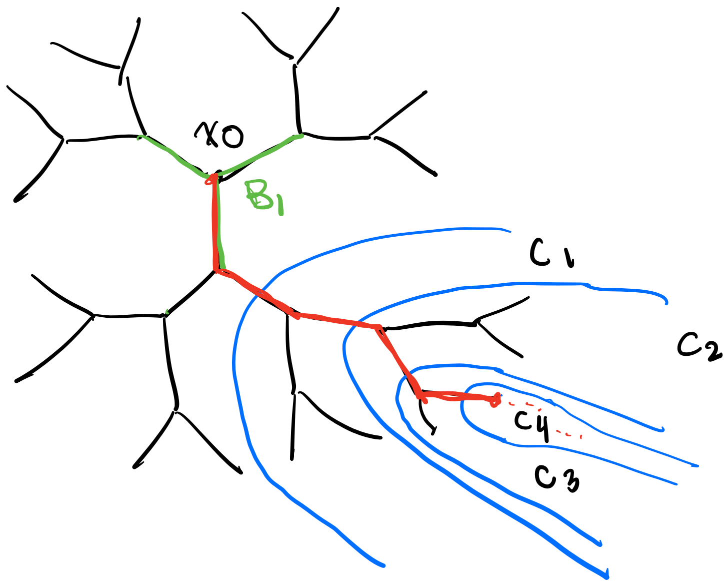
$\Rightarrow \quad X \setminus B_n(x_0) \supseteq X \setminus B_m(x_0')$

$\Rightarrow \quad \exists!$ component $C_n \subseteq X \setminus B_n(x_0)$ containing C_m' for all $m \geq r_n$

So the end $\varepsilon = C_1 \supset C_2 \supset \dots$ is determined by $C_1' \supset C_2' \supset \dots$

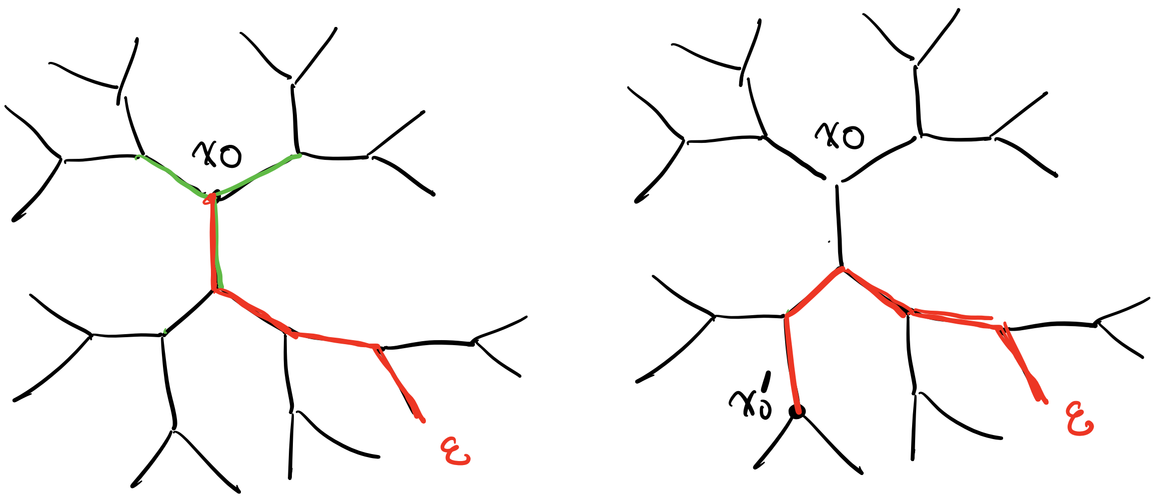


Example: $X = T_3$ = infinite trivalent tree



$C_1 \supset C_2 \supset C_3 \supset \cdots \quad \longleftrightarrow \quad$ a geodesic ray, going out to ∞ (the red path)

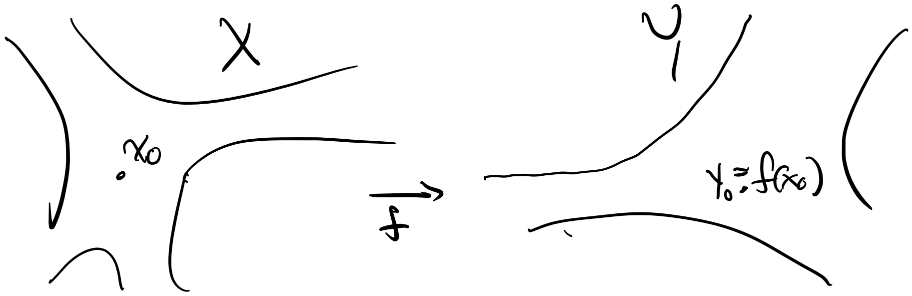
T_3 has infinitely many ends



Different geodesic rays, but same end ε

Theorem Let X be a proper geodesic metric space. The number of ends of X is a quasi-isometry invariant.

Proof Given a q.i. $X \xrightarrow{f} Y$,



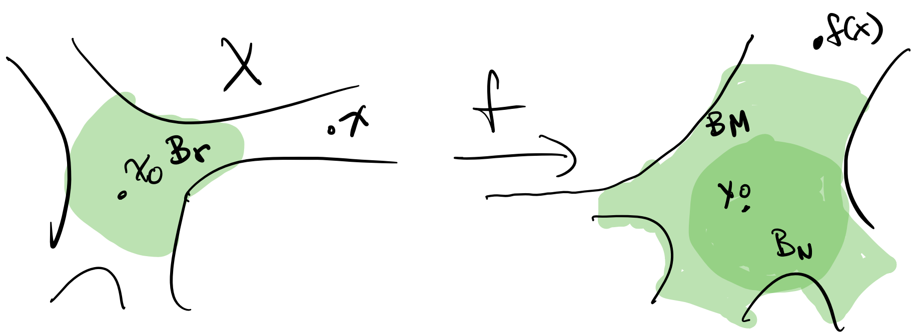
$$\frac{1}{\lambda} d_X(x,x') - C \le d_Y(f(x), f(x')) \le \lambda d_X(x,x') + C$$

Fix $N > 0$ pick $x_0 \in X$, let $y_0 = f(x_0)$

and take $M > N + C$

Then

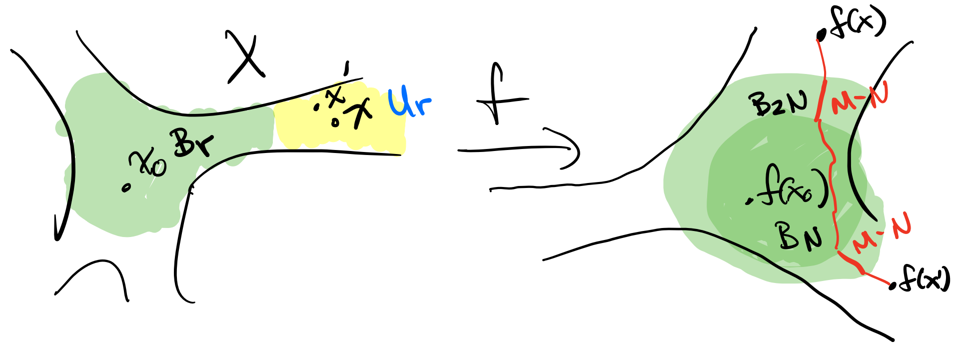
① $d(x, x_0) > r = \lambda(M+C) \Rightarrow d(f(x), y_0) \ge \frac{1}{\lambda} d(x,x_0) - C > M$



and ② $d(x,x') < \varepsilon = \frac{M-N-C}{\lambda}$ $(\varepsilon > 0)$

$\Rightarrow d(f(x), f(x')) < M - N$

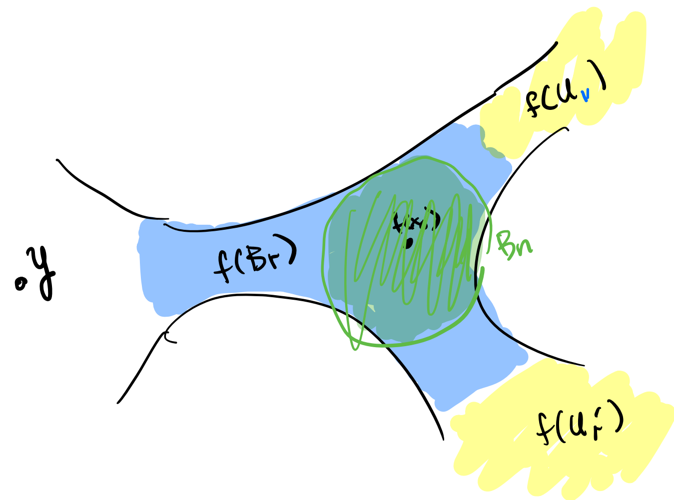
Now suppose x, x' are in the same component of $X \setminus \overline{B}_r$
and $d(x,x') < \varepsilon$.
If $f(x)$ and $f(x')$ are in different components of
$Y \setminus \overline{B}_{\underline{\underline{N}}}(y_0)$ then $d_Y(f(x), f(x')) > 2(M-N)$ $*$



So: If x, y are in the same component U_r of $X \setminus \overline{B}_r$,
connect them by a path, divide the path into pieces of size $< \varepsilon$, conclude $f(x), f(y)$ are in the same component of $Y \setminus B_N$.

Now $f(B_r) \subset B_{\lambda r + c}(y_0)$, so we can conclude
$\text{ends}(X) \longrightarrow \text{ends}(Y)$ is surjective:

If not, can find a point y in some end of Y that is arbitrarily far from $f(X)$, contradicting the fact that f is quasi-surjective.



So: If # ends is finite, they are the same.
Otherwise they are both infinite.
(They have the same cardinality)

Cor $F_n \not\sim_{qi} \mathbb{Z}$: if $n \geq 2$,

in fact $F_n \not\sim \mathbb{Z}^k$ if $n \geq 2$, $k \geq 2$

Pf $F_n \sim_{qi} T_3$ which has infinitely many ends
$\mathbb{Z} \sim_{qi} \mathbb{R}$ which has two ends
$\mathbb{Z}^k \sim_{qi} \mathbb{R}^k$ which has one end if $k \geq 2$

Q: is $\mathbb{Z}^2 \sim_{qi} \mathbb{Z}^3$? (A: no, but you can't tell by counting ends!)

Theorem: (Stallings) A finitely generated group G has $0, 1, 2$ or infinitely many ends

<u>proof</u>: Suppose $\mathcal{G} = \mathcal{G}(G,S)$ has $2 < e < \infty$ ends

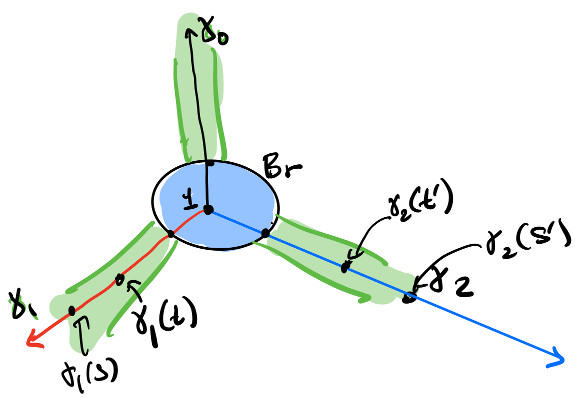
$g \in G \Rightarrow g : \mathcal{G} \to \mathcal{G}$ is an isometry,
so permutes the finite number of ends

Let $N = \operatorname{Ker}\left(\begin{array}{ccc} G & \longrightarrow & \Sigma_{(\text{ends})} \\ g & \longmapsto & \sigma_g \end{array} \right)$

N fixes the ends and has finite index in G
so is quasi-isometric to G; in particular
N has the same number of ends as G.
ie we may assume G fixes the ends of $\mathcal{G}$.

Take r big enough so $\mathcal{G} \setminus B_r$ has ≥ 3 components

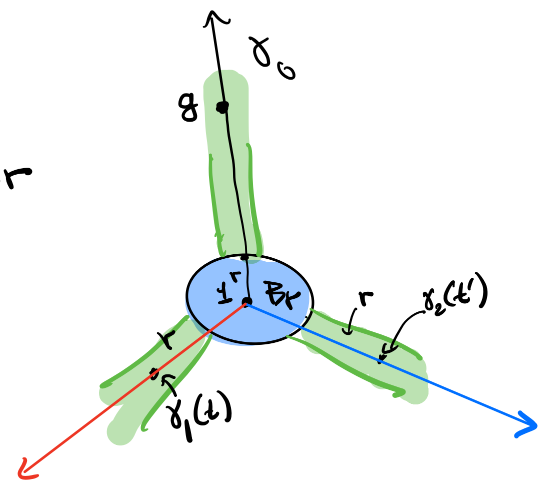
Take $\gamma_0, \gamma_1, \gamma_2$ geodesics going far into each end



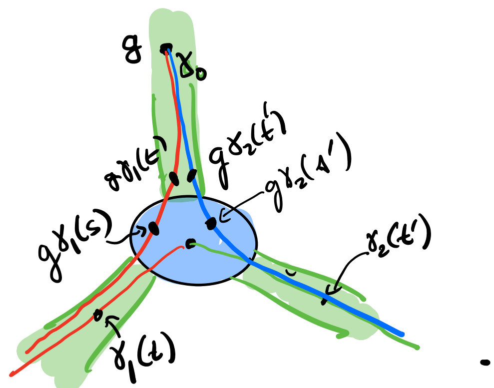
Let t, t' be paramaters with
$d(\gamma_1(t), 1) = 2r$
$d(\gamma_2(t'), 1) = 2r$

If $s > t$, $s' > t'$, then $d(\gamma_1(s), \gamma_2(s')) > 2r$

Take $g \in \gamma_0$ with $d(1,g) = 3r$



Apply g to γ_1 and γ_2



There is $s > t$ with $g\gamma_1(s) \in B_r$.

There is $s' > t'$ with $g\gamma_2(s') \in B_r$.

so $d(g\gamma_1(s), g\gamma_2(s')) < 2r$

But $d(g\gamma_1(s), g\gamma_2(s')) = d(\gamma_1(s), \gamma_2(s')) > 2r$ $\star$.

Presentations and Cayley complexes.

Presentations. Another way to say G is generated by $S \subset G$ is that the homomorphism $f: F(S) \longrightarrow G$ extending the inclusion $S \hookrightarrow G$ (which exists by the universal property) is onto. Then the 1st isomorphism theorem says

$$G \cong F(S)/N,$$

where $N = \ker f < F(S)$.

N is a normal subgroup, ie if $w \in N$ and $u \in F(S)$ then $uwu^{-1} \in N$

<u>Definition</u>: N is normally generated by $R \subset F(S)$ if R and all conjugates of elements of R generate N.

Write $N = \langle\langle R \rangle\rangle$. If N is normally generated by R, then S and R completely determine G

We write $G = \langle S \mid R \rangle$; this is called a presentation for G.

Remarks:

- Every group has a presentation: take $S = G$, R = all words in $\ker(F(S) \to G)$
- Groups have lots of different presentations:
  - different generating sets for G
  - different normal gen sets for N

Definition G is finitely presented if $G = \langle S \mid R \rangle$ with $|S| < \infty$ and $|R| < \infty$.

examples:

$$F(S) = \langle S \mid \ \rangle$$

$$C_n = \langle t \mid t^n \rangle$$

$$\mathbb{Z}^2 = \langle e_1, e_2 \mid e_1 e_2 e_1^{-1} e_2^{-1} \rangle$$

Notation: $e_1 e_2 e_1^{-1} e_2^{-1} = [e_1, e_2]$

$$\mathbb{Z}^n = \langle e_1, \dots, e_n \mid [e_i, e_j] \text{ for all } i,j \rangle$$

$$1 = \langle a \mid a \rangle$$

A presentation completely determines a group, but it is not always easy to tell <u>which</u> group

<u>Example</u>:

$$\langle a, b \mid a^{-1}bab^{-2},\ b^{-1}aba^{-2} \rangle \overset{?}{=} 1 \quad (!)$$

Dehn's isomorphism problem:

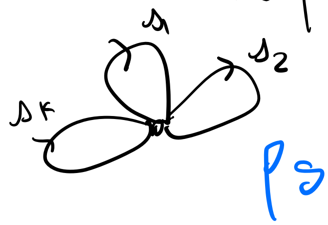
Is there an algorithm to decide whether two finite presentations give isomorphic groups?

(Answer: NO - there is no algorithm that can decide for all pairs of finite presentations.

But if you have more information about the groups, there may be an algorithm.)

Cayley complexes

Suppose G has a finite presentation $\langle S \mid R \rangle$.
Let ρ_S be the oriented rose with one petal for each $s \in S$:



Van Kampen $\Rightarrow$ attaching a 2-cell with boundary w to the rose kills w in π_1, so also kills all conjugates of w

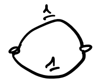
Let $X = X(G,S,R)$ be the space obtained by attaching a 2-cell to ρ_S for each $w \in R$.
The map induced by inclusion $\rho_S \hookrightarrow X$

$$F(S) = \pi_1(\rho_S) \xrightarrow{i_*} \pi_1(X)$$

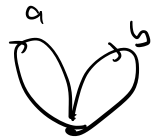
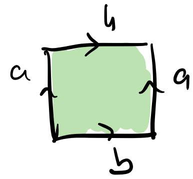
is surjective and it's kernel is normally generated by R, ie $\pi_1(X) = G$

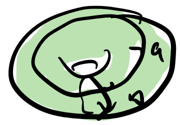
The universal cover $\tilde{X}$ is the Cayley complex $\mathcal{C}(G,S,R)$

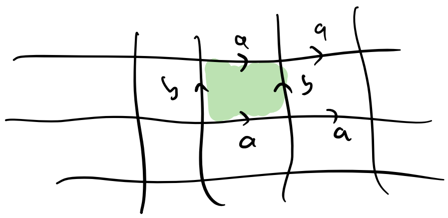
It can be made into a proper geodesic metric space by making each lift of a 2-cell killing w of length k isometric to a regular Euclidean k-gon with side length 1 (if $k=2$, use a disk with circumference 2)



eg: $G = \mathbb{Z}^2$, $S = \{a, b\}$ $R = \{aba^{-1}b^{-1}\}$

$X(G,S,R) =$  $\cup$ 

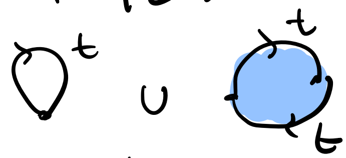
$\cong$  $\pi_1 \cong \mathbb{Z}^2$

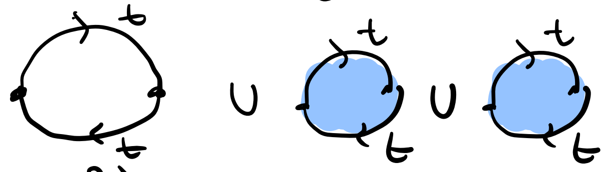
$\mathcal{C}(G,S,R) =$ 

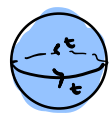
G acts freely and cocompactly by isometries on $\mathcal{C}(G,S,R)$, so by Svarc-Milnor, $\mathcal{C}(G,S,R)$ is quasi-isometric to G

The 1-skeleton of $\mathcal{C}(G,S,R)$ is the Cayley graph $\mathcal{C}(G,S)$.

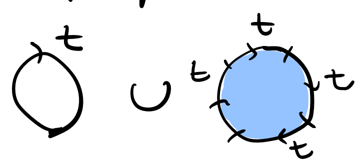
Example: $G = \mathbb{Z}/2 = \langle t \mid t^2 \rangle$

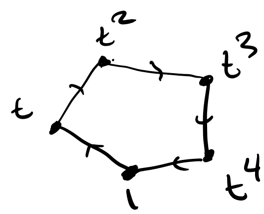
Presentation complex:  $(= \mathbb{R}P^2)$

Universal cover: 

$=$  $(= S^2)$

Example: $G = \mathbb{Z}/n\mathbb{Z} = \langle t \mid t^n \rangle$

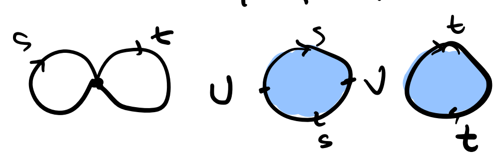
Presentation complex $=$ 

Universal cover: $(n=5)$ 

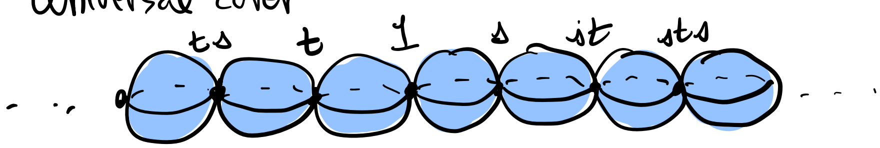
$+5$ 2-cells, each with the same ∂.

$=$ Cayley complex.

Example: $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle s,t \mid s^2, t^2 \rangle$

Presentation complex 

Universal cover



$=$ Cayley complex.

Advantages of $\mathcal{C}(G,S,R)$ over $\mathcal{C}(G,S)$:

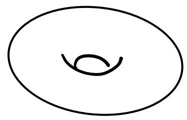
$\mathcal{C}(G,S,R)$ together with its action of G contains all information about G, since you can recover G from it ($G = \pi_1(G \backslash \mathcal{C}(G,S,R))$).

whereas you can only recover quasi-isometry invariants of G from $\mathcal{C}(G,S)$ and its action by G.

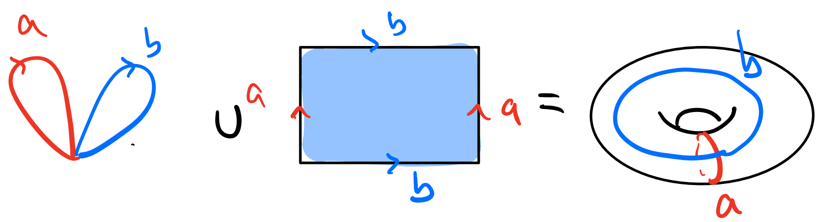
Surface groups

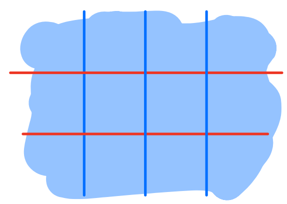
$\Sigma_g =$ Closed orientable surface of genus $g \geq 1$

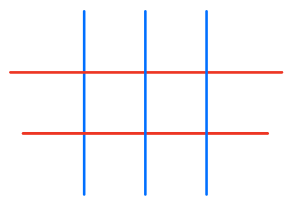
$\pi_1(\Sigma_g)$ is called a **surface group**

$g=1$ $\quad \Sigma_1 = T^2 = S^1 \times S^1 =$ 

$\pi_1 \Sigma_1 = \mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

Presentation complex $=$ 

Cayley complex $=$ 

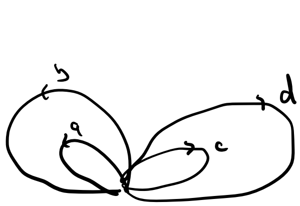
Cayley graph $=$ 

Cayley cplex is tiled by squares of side length 1, and is isometric to $\mathbb{R}^2$.

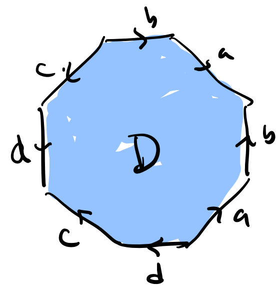
The same topological picture holds for surfaces of genus ≥ 2.

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1 \rangle$$

Eg $g=2$ $\quad \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

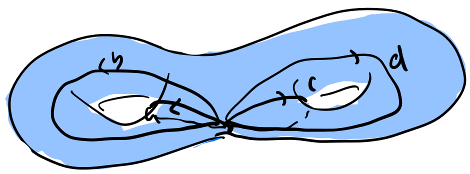


$\cup$



$\parallel$

$\Sigma_2 =$



so the Cayley complex $\widetilde{\Sigma}_2$ is tiled by octagons, 8 of them meet at every vertex, and $\pi_1\Sigma_2$ acts by translating them around.

$\widetilde{\Sigma}_2$ is a simply-connected surface, so is homeomorphic to $\mathbb{R}^2$ or S^2.

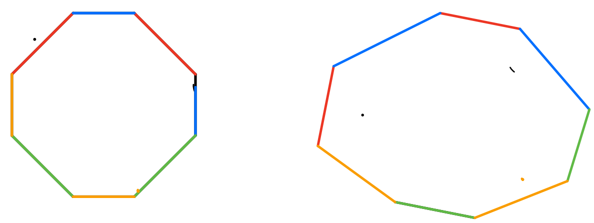
$(\pi_1\Sigma_2)^{ab} = \mathbb{Z}^{2g}$, so $\pi_1\Sigma_2$ is infinite.

Since it acts freely and properly on $\widetilde{\Sigma}_2$, $\widetilde{\Sigma}_2$ can't be S^2.

So, $\tilde{\Sigma}_2$ is homeomorphic to $\mathbb{R}^2$.

But you can't make it isometric to $\mathbb{R}^2$: you can't tile $\mathbb{R}^2$ with isometric octagons (of *any* shape!)

Σ angles $= 6\pi$

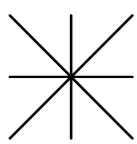


But you *can* tile the hyperbolic plane by isometric octagons — by many *different* shaped octagons, in fact

(And there is nothing special about octagons — you can tile $\mathbb{H}^2$ by n-gons for any $n \geq 5$.)

For example you can tile $\mathbb{H}^2$ by regular octagons

(all sides have same length, all interior angles are $\frac{\pi}{4}$).



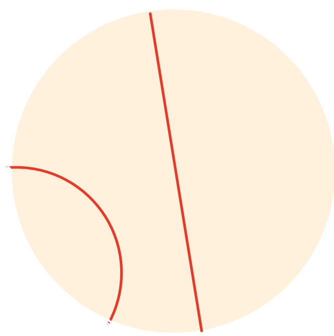
So Σ angles at any vertex $= 2\pi$.

We talked about the action of $SL_2\mathbb{Z}$ on the upper half-space $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.
With the metric $\frac{ds}{y} = \sqrt{dx^2+dy^2}/y$ this is one model for the hyperbolic plane.
With this metric the action of $SL_2\mathbb{Z}$ is by isometries.
(called Möbius transformations)
But to see the tiling of $\tilde{\Sigma}_g$ by $4g$-gons, it is more convenient to use the Poincaré disk model $\mathbb{D}$. This is the interior of the unit disk in $\mathbb{R}^2$ with the metric $\frac{2ds}{1-r^2}$. The map

$$f : \left(\mathbb{H}, \frac{ds}{s}\right) \longrightarrow \left(\mathbb{D}, \frac{ds}{1-r^2}\right)$$
$$z \longmapsto \frac{z-i}{z+i}$$

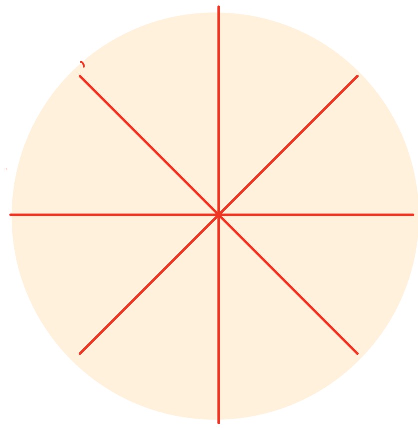
is an isometry.

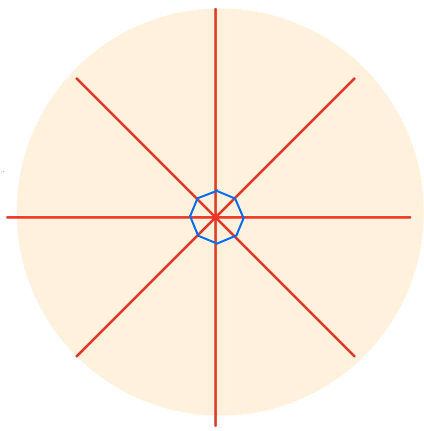
Geodesics in $\mathbb{D}$ are circle-arcs $\perp$ to S^1 and straight lines through the origin



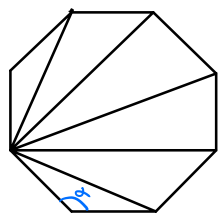
Isometries are generated by inversions in circles, and there's an orientation-preserving isometry taking any geodesic to any other geodesic, and any point to any other point.

To find a regular octagon in $\mathbb{D}$ with all angles $\frac{\pi}{4}$, take points on these rays: at distance r from O

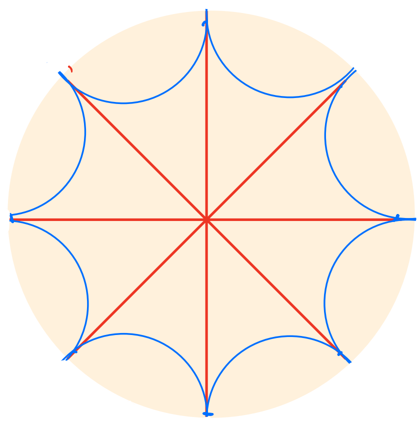




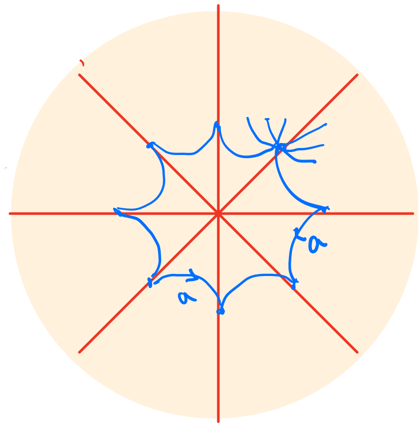
If r is small, $\frac{ds}{1-r^2}$ is almost ds, the octagon is almost Euclidean, so has angles almost $\frac{3\pi}{4}$



(in $\mathbb{R}^2$, $6\cdot\pi = 8\cdot\alpha$, so $\alpha = \frac{3\pi}{4}$)



If r is close to 1, the angles are almost 0



So somewhere in between they are $\frac{\pi}{4}$. So 8 of them fit around each vertex (see picture on last page of notes) and translates tile $\mathbb{D}$, ie $\widetilde{\Sigma}_2$ can be identified with $\mathbb{D}$.

$\pi_1 \Sigma_2$ acts freely by translating the octagons around, and translates cover $\mathbb{D}$, so the action is cocompact.
Therefore by Švarc-Milnor, $\pi_1 \Sigma_g$ is quasi-isometric to $\mathbb{D}$, ie distances in $\pi_1 \Sigma_g$ (with any word metric can be approximated by distances in $\mathbb{D}$.

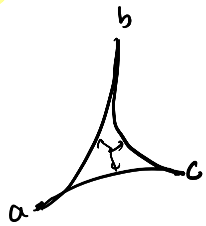
Max Dehn used intuition from this picture to answer some fundamental questions about surface groups.
For example:
The hyperbolic plane has a linear isoperimetric function... the minimal area needed to fill a simple closed loop of length n with a disk is linear in n
Dehn found an algorithm to decide whether a word in the standard generators is trivial, and showed the number of relators you need to use to prove a word is trivial is linear in the length of the word.

He also found algorithms to decide whether two words were conjugate, and whether two presentations of surface groups determined the same surface group.

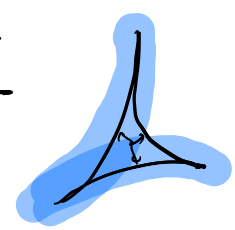
We are interested in classifying groups up to quasi-isometry

Gromov singled out certain geometric features of $\mathbb{D}$ and proved they were quasi-isometry invariants.
He proved that groups with these features satisfy many of the same properties as surface groups, and called them hyperbolic groups

These features include
① geodesic triangles are "thin"

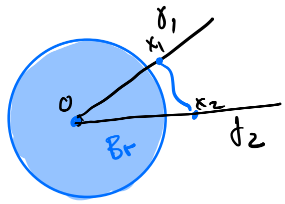


There are many ways to say this. One way:
there is a constant δ $(= \log 3)$ such that each side is contained in a δ-neighborhood of the other 2 sides:



The triangle is "δ-thin"

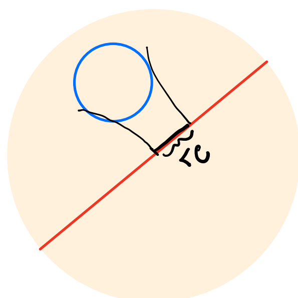
② Geodesics diverge exponentially fast, i.e.



There is a constant $C>0$ satisfying:
If $x_i \in \gamma_i$, $d(x_i, 0) > r$ let $d_r(x_1, x_2)$ be the length of the shortest path from x_1 to x_2 that stays outside $B_r(0)$.

then $d_r(x_1, x_2) \geq C^r$

③ There is a constant $C>0$ satisfying:
If a ball in $\mathbb{D}$ is disjoint from a geodesic, its projection onto the geodesic has length $\leq C$.



It turns out that any proper geodesic metric space satisfying ① also satisfies ② and ③, so we will use ① to define the notion of "hyperbolic metric space".