Gromov hyperbolic metric spaces

Definition: Let $\delta > 0$. A metric space X is δ-hyperbolic if geodesic triangles are δ-thin: each side is contained in a δ-neighborhood of the other two sides.

A metric space is (Gromov) hyperbolic if it is δ-hyperbolic for some $\delta > 0$.

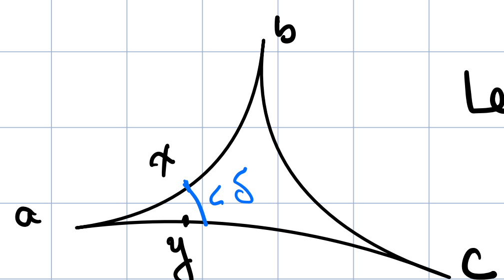
From now on we will assume all metric spaces are proper geodesic metric spaces unless otherwise noted.

Notation: $[x,y]$ means a geodesic from x to y

Geodesic triangles in a δ-hyperbolic space

Let Δ be a geodesic triangle with vertices a, b, c

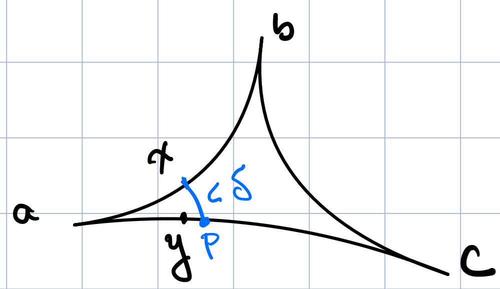
① Let $x \in [a,b]$. Then $d(x,[a,c]) < \delta$ or $d(x,[b,c]) < \delta$; assume the former.



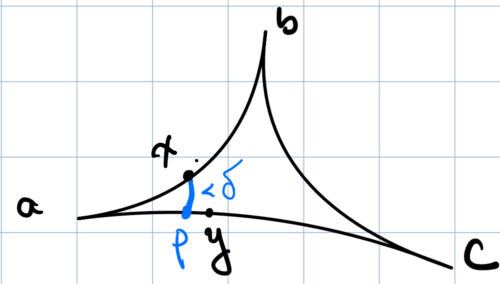
Let $y \in [a,c]$ with $d(a,x) = d(a,y)$

Then $d(x,y) < 2\delta$

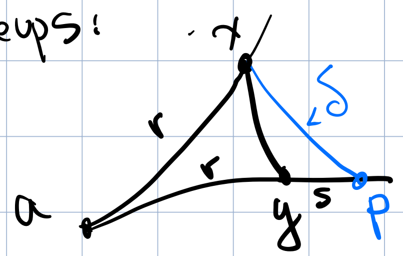
proof: 2 cases



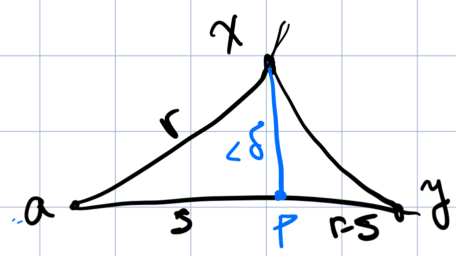
and



closeups:



and



$r+s \le r+\delta \Rightarrow s \le \delta$

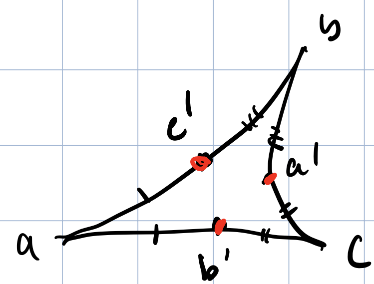
$\Rightarrow d(x,y) \le s+\delta \le 2\delta$ ✓

$r \le s+\delta \Rightarrow r-s \le \delta$

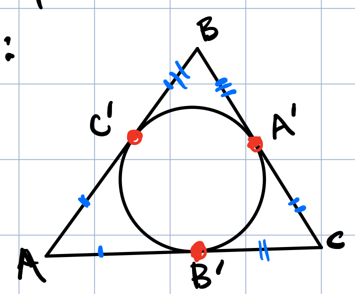
$\Rightarrow d(x,y) \le \delta + (r-s) \le 2\delta$ ✓

(2) The internal points of the triangle are the points $a' \in [b,c]$, $b' \in [a,c]$ and $c' \in [a,b]$ with

- $d(a,c') = d(a,b')$
- $d(b,c') = d(b,a')$
- $d(c,a') = d(c,b')$



(To see they exist, consider a triangle in $\mathbb{R}^2$ with the same side lengths, then draw the largest inscribed circle:

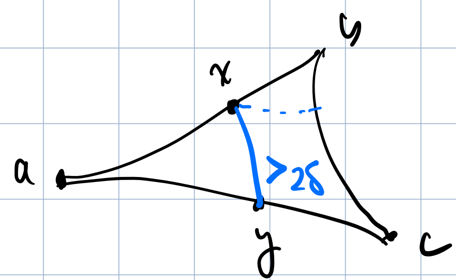


)

Then $d(a',c')$, $d(b',c')$ and $d(b',c')$ are all $< 4\delta$ (two of them are $< 2\delta$!)

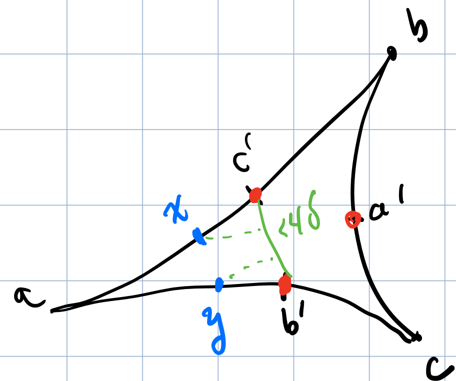
(3) If $x \in [a,b]$ and $y \in [a,c]$ with $d(a,x) = d(a,y)$ and $d(x,y) > 2\delta$

then $d(x,[b,c]) < \delta$ and $d(y,[b,c]) < \delta$.



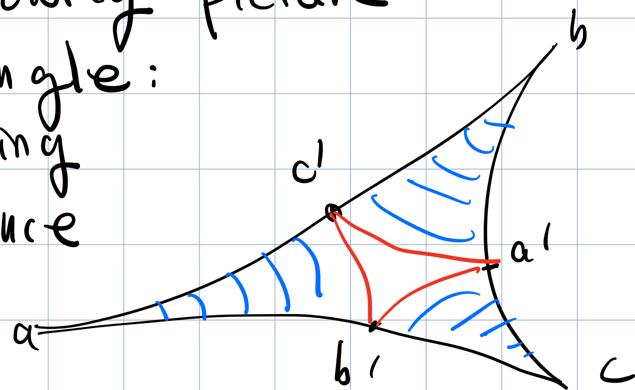
(4) If $x \in [a,c']$, $y \in [a,b']$, and $d(a,x) = d(a,y)$

Then $d(x,y) < 6\delta$.



So we have the following picture of a geodesic triangle:

Draw geodesics joining points at equal distance from corners:



The blue geodesics have length $< 6\delta$

The red geodesics have length $< 4\delta$.

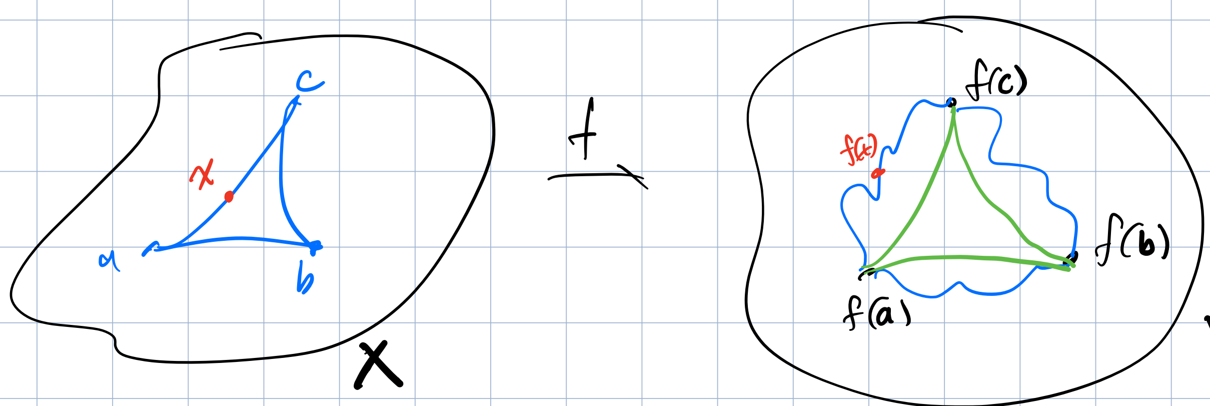
Theorem Let $f: X \to Y$ be a quasi-isometry between proper geodesic metric spaces. If Y is hyperbolic, then so is X.

Proof Let λ, C, K be the constants for f and suppose Y is δ-hyperbolic.

Let Δ be a geodesic triangle in X, with vertices a, b, c. We want to show it is δ'-thin for some δ'
(δ' will depend on δ, λ, C, K.)

1a. For $x \in [a,c]$ we want to find $y \in [a,b]$ or $[b,c]$ with $d(x,y) < \delta'$

Look at $f(\Delta) \subset Y$

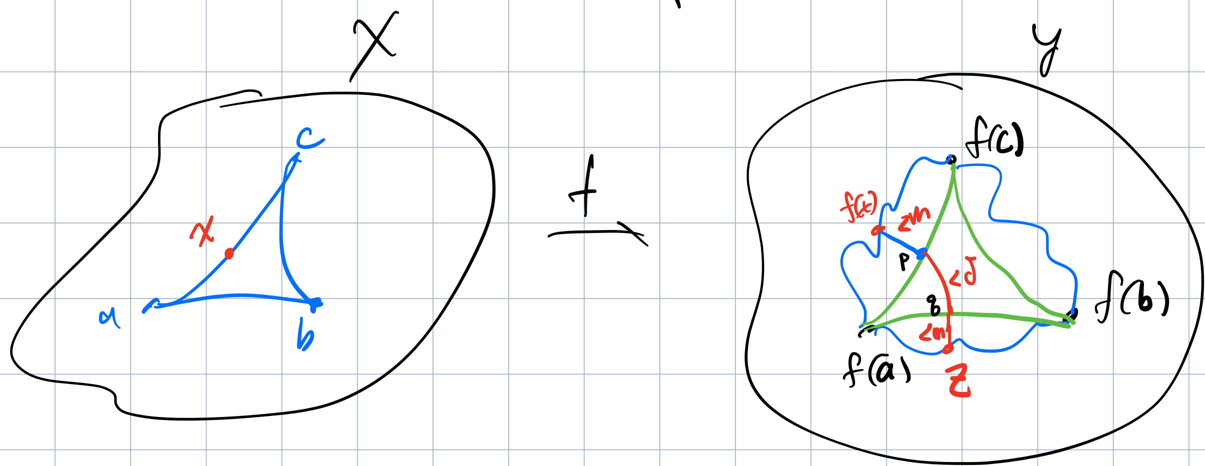


There is a (green) geodesic triangle in Y with vertices $f(a)$, $f(b)$, $f(c)$.

Suppose we know that $f[a,b]$ and $[f(a),f(b)]$ stay a bounded distance apart, ie

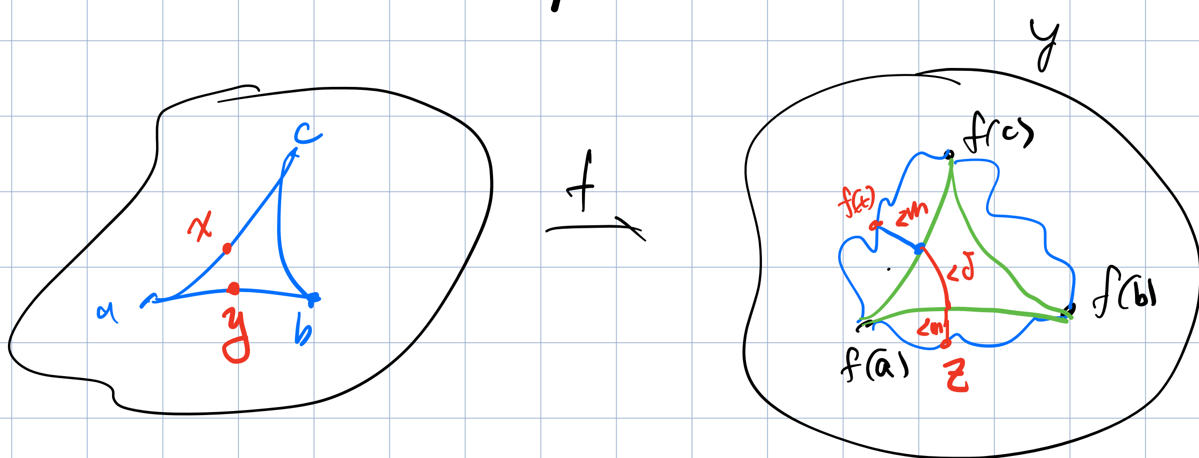
$$(*)\quad \boxed{\begin{array}{l}\exists\, m=m(\delta) \text{ s.t. } \forall\, a,b \;\; f[a,b]\subset N_m[f(a),f(b)] \\ \quad \underline{\text{and}}\;\; [f(a),f(b)]\subset N_m(f[a,b])\end{array}}$$

Then we can finish the proof:



Find $p\in[f(a),f(b)]$ w/ $d(f(x),p)<m$. Then find $q\in[f(a),f(b)]$ w/ $d(p,q)<\delta$. Then find z on $f[a,b]$ with $d(q,z)<m$. So altogether, $d(f(x),z)<\delta+2m$

Now $z=f(y)$ for some y:



so $\delta+2m \geq d(f(x),f(y)) \geq \frac{1}{\lambda}d(x,y)-c$

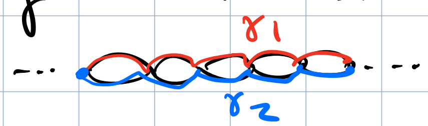
ie $d(x,y)\leq \delta' = \lambda(\delta+2m+c)$

This argument depends on showing (*)!

To prove (*) we will first show that geodesics diverge exponentially fast.

But they don't necessarily!

eg



However, if they get sufficiently far apart they do:

<u>Theorem</u> Let X be δ-hyperbolic, $x \in X$ and γ_1, γ_2 geodesics starting at x, paramaterized by arc length.

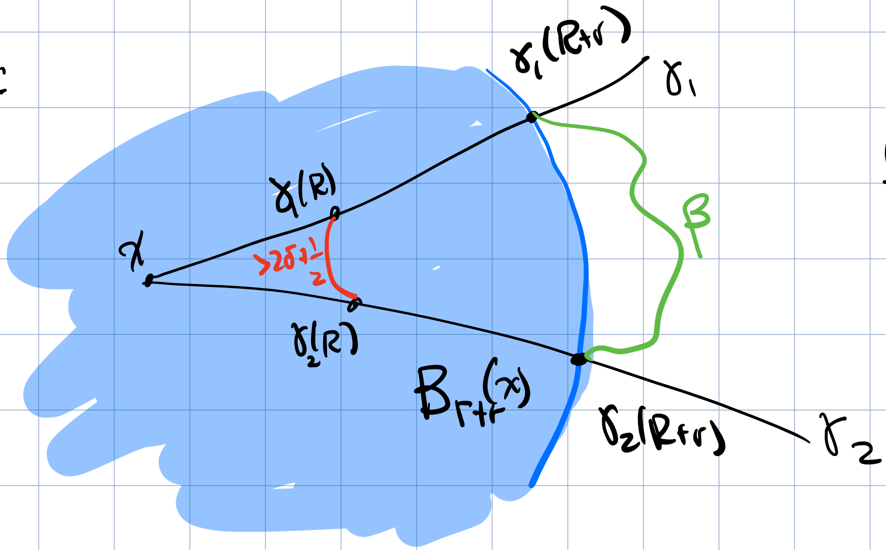
Suppose that $d(\gamma_1(R), \gamma_2(R)) \geq 2\delta + \frac{1}{2}$ for some $R > 0$.

Then there are constants K, μ such that for all $r>0$,

$$d_{R+r}(\gamma_1(R+r), \gamma_2(R+r)) > Ke^{\mu r},$$

where $d_N(p,q)$ is the length of the shortest path joining p and q that stays outside $B_N(x)$

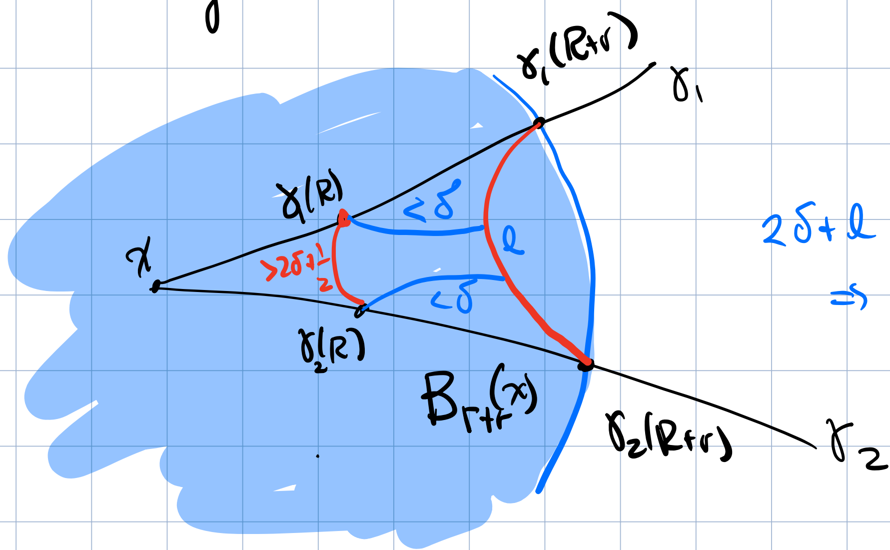
Picture:



$\ell(\beta) > Ke^{\mu r}$.

Proof:

Since $d(\gamma_1(R), \gamma_2(R)) > 2\delta$, we know both $\gamma_1(R)$ and $\gamma_2(R)$ are within δ of the geodesic from $\gamma_1(R+r)$ to $\gamma_2(R+r)$



$2\delta + \ell \geq 2\delta + 1/2$

$\Rightarrow \ell \geq 1/2$

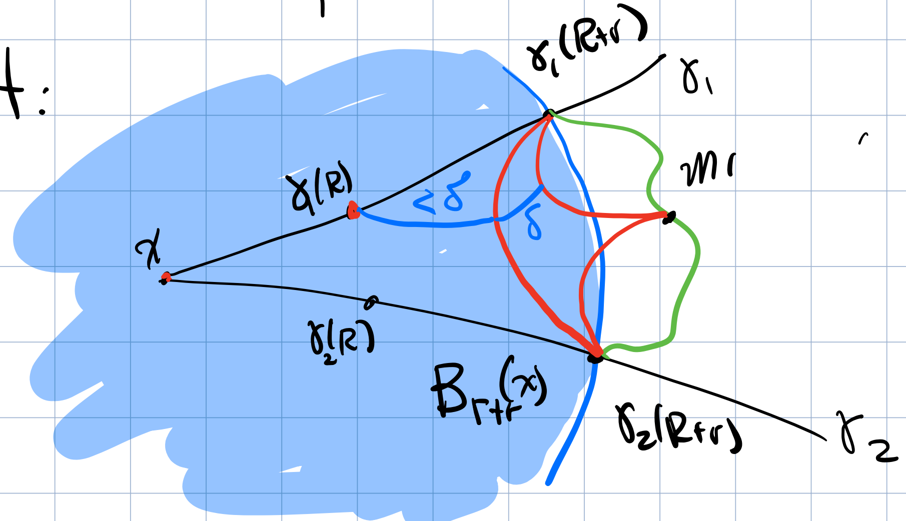
This in turn implies

$$d(\gamma_1(R+r), \gamma_2(R+r)) \geq \frac{1}{2}$$

So there is some $n \geq 0$ s.t. $2^{n-1} < \ell(\beta) < 2^n$

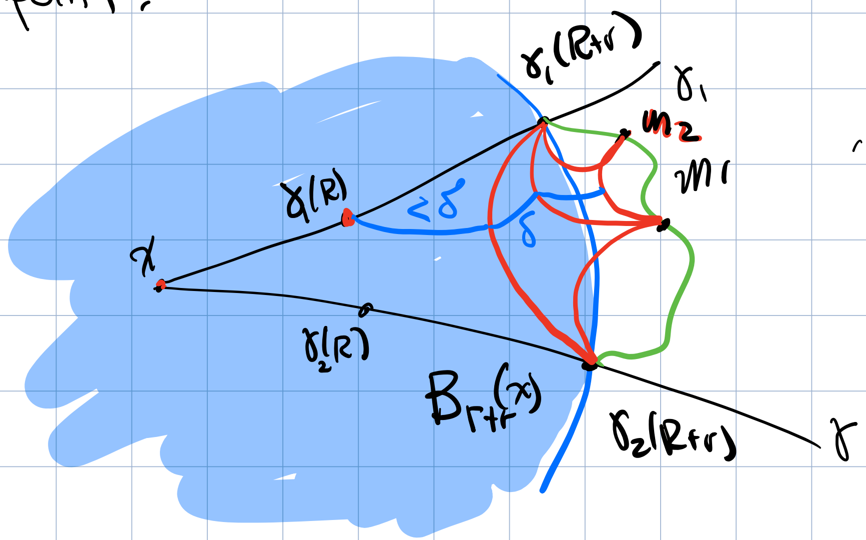
If we take midpoints n times we cut β into 2^n pieces, each of length between $\frac{1}{2}$ and n. We will use the fact that all the midpoints are outside $B_{R+r}(x)$ to estimate n, and therefore $\ell(\beta)$:

First midpoint:



$$R+r < d(x, m_1) < R+2\delta + d(\gamma_1(R+r), m_1)$$
$$\leq R+2\delta + \frac{\ell(\beta)}{2}$$

Next midpoint:



$$R+r < d(x, m_2) < R+3\delta + d(m_1, m_2) \leq R+3\delta + \frac{\ell(\beta)}{4}$$

etc. after n midpoints you get

$$\cancel{R}+r < d(x, m_n) < R + (n+1)\delta + \frac{\ell(\beta)}{2^n}$$

$$< \cancel{R} + (n+1)\delta + 1$$

so $\quad r < (n+1)\delta + 1$,

so $\quad n-1 > \frac{r}{\delta} - \frac{1}{\delta} - 2$

Then $\quad \ell(\beta) > 2^{n-1} > \underbrace{2^{\frac{r}{\delta}}}_{e^{r\mu}} \cdot \underbrace{2^{-\frac{1}{\delta}-2}}_{K} = e^{r\mu} \cdot K$ ✓

<u>Definition</u>: Let $I \subset \mathbb{R}$ be a closed interval.
A quasi-geodesic embedding $\alpha: I \longrightarrow X$ is called a quasi-geodesic.

ie $\exists\, \lambda, C$ s.t.

$$\frac{1}{\lambda}|r-s| - C \leq d(\alpha(r), \alpha(s)) \leq \lambda|r-s| + C$$

If $[a,b]$ is an edge of our triangle in X, and $\gamma: I \longrightarrow [a,b]$ a paramatrization by arc length, then $\alpha = f\circ\gamma: I \longrightarrow X$ is a quasi-geodesic with image $f[a,b]$.

So to show $(*)$ we want to show "quasi-geodesics stay close to geodesics"

To prove this, it is convenient to assume quasi-geodesics are continuous. But quasi-isometric embeddings are <u>not</u> necessarily continuous!

We can get around this by proving that there is a <u>continuous</u> quasi-geodesic β that stays close to α, ie

Lemma: Given $\alpha : [a,b] \to X$ a (λ, C)-quasi-geodesic. Then there is a continuous $(\lambda, 2(\lambda+C))$-quasi-geodesic $\beta : [a,b] \to X$ s.t.

① $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$

② $\beta \subset N_{\lambda+C}(\alpha)$, $\alpha \subset N_{\lambda+C}(\beta)$

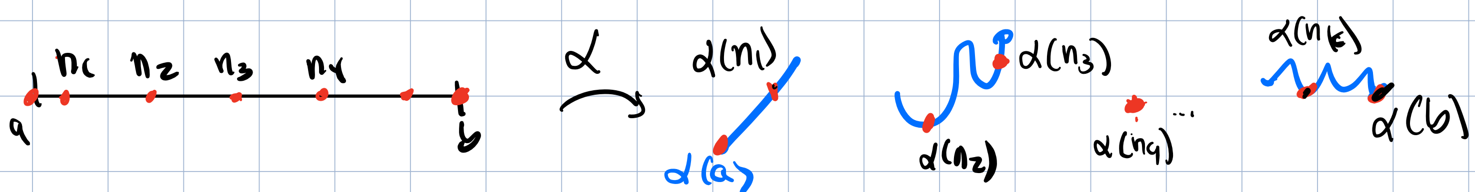
and

③ If $s,t \in [a,b]$ then

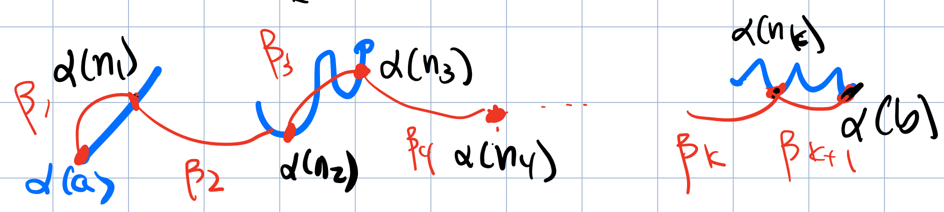
$$\ell(\beta|_{[s,t]}) \leq \lambda' d(\beta(s), \beta(t)) + C'$$

for constants λ', C' depending on δ, λ, C

Proof: Let $n_1 < n_2 < \dots < n_k$ be the integer points in $[a,b]$:



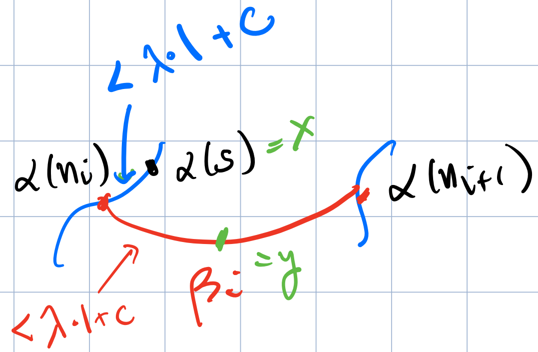
Connect $\alpha(a)$ to $\alpha(n_1)$, $\alpha(n_i)$ to $\alpha(n_{i+1})$ and $\alpha(n_k)$ to $\alpha(b)$ by geodesics $\beta_1, \dots, \beta_{k+1}$:



Then $\ell(\beta_i) \le d(\alpha(n_{i-1}), \alpha(n_i)) \le \lambda \cdot 1 + C$ for all i.

So $\beta \subseteq N_{\lambda+C}(\alpha)$

and $\alpha \subseteq N_{\lambda+C}(\beta)$



To see that β is a quasi-geodesic, check the inequalities:

Note $d(\alpha(s), \beta(s)) \le 2(\lambda + C)$ $\forall s$.

So $d(\beta(s), \beta(t)) \le d(\beta(s), \alpha(s)) + d(\alpha(s), \alpha(t)) + d(\alpha(t), \beta(t))$

$$\le 4(\lambda + C) + d(\alpha(s), \alpha(t))$$

$$\le 4(\lambda + C) + \lambda |s-t| + C = \lambda |s-t| + C'$$

and $|s-t| \le \lambda\, d(\alpha(s), \alpha(t)) + C\lambda$

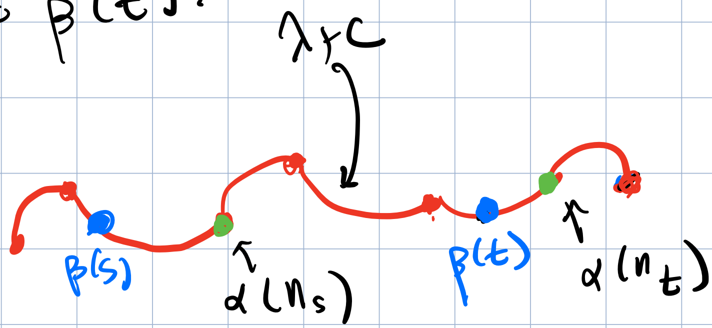
$$\le \lambda [d(\alpha(s), \beta(s)) + d(\beta(s), \beta(t)) + d(\beta(t), \alpha(t))] + C\lambda$$

$$\le \lambda (4(\lambda + C) + d(\beta(s), \beta(t))) + C\lambda$$

so $d(\beta(s), \beta(t)) \ge \dfrac{|s-t| - C\lambda}{\lambda} - 4(\lambda + C)$

$$= \frac{1}{\lambda} |s-t| - C''$$

It remains to prove ③, ie we want to bound the length of β from $\beta(s)$ to $\beta(t)$ in terms of the distance from $\beta(s)$ to $\beta(t)$:



$$\ell(\beta|_{[s,t]}) \leq (\lambda+C)(n_t - n_s) + 2(\lambda+C)$$

$$\leq (\lambda+C)(t-s+3)$$

$$\leq (\lambda+C)\left(\frac{1}{\lambda'} d(\beta(s),\beta(t)) - C'\right) + 3(\lambda+C)$$

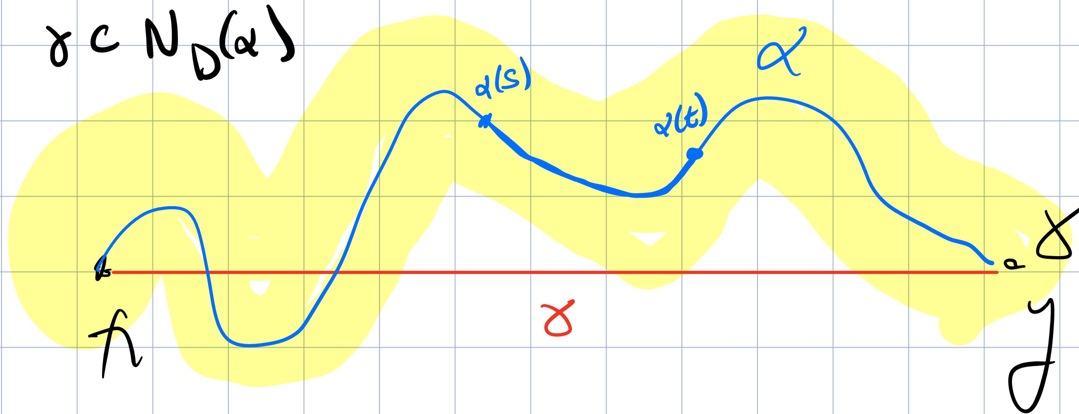
$$= \left(\frac{\lambda+C}{\lambda'}\right) d(\beta(s),\beta(t)) - (\lambda+C)C' + 3(\lambda+C)$$

$$= \lambda'' d(\beta(s),\beta(t)) + C''$$

Theorem: X a δ-hyperbolic space
$x, y \in X$, α a (λ, c)-quasi-geodesic
$\alpha(a) = x$, $\alpha(b) = y$, γ a geodesic from x to y.
$\therefore$ Then there is $D = D(\lambda, c)$ s.t.
① $\gamma \subset N_D(\alpha)$ and ② $\alpha \subset N_D(\gamma)$

$\underline{\text{pf}}$: Lemma shows we may assume α is continuous and
$$\ell(\alpha|_{[s,t]}) \leq \lambda\, d(\alpha(s), \alpha(t)) + C$$

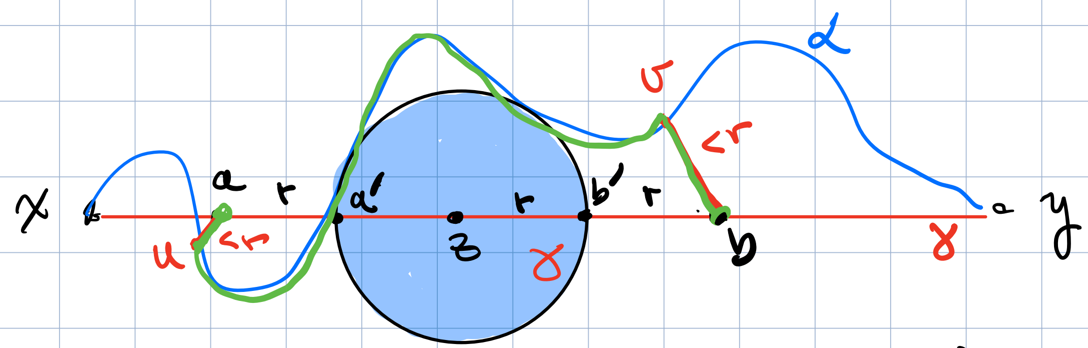
① $\gamma \subset N_D(\alpha)$



Idea: Find $z \in \gamma$ as far as possible from α, say distance r, show r is bounded by a fcn $D = D(\lambda, C)$

Details:

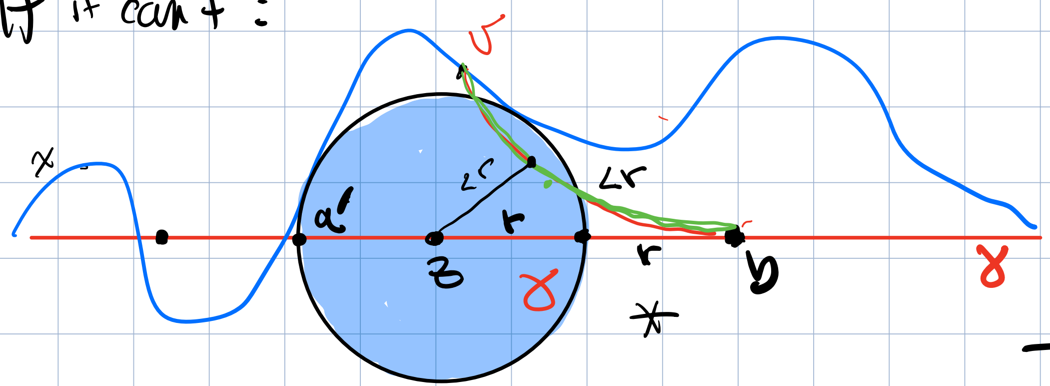
Find $z \in \gamma$ with largest possible ball $B_r(z)$ disjoint from α:



The green path stays outside $B_r(z)$:

[you might worry the geodesic from b to v enters $B_r(z)$

But it can't:



]

Now u, v are on α, so length of green path from u to v is

$$\leq \lambda\, d_X(u,v) + C \leq \lambda \cdot 6r + C$$

(since $d_X(u,v) \leq 6r$)

So the length of the entire green path is $\leq \lambda \cdot 6r + C + 2r$.

But the green path has length $> Ke^{\mu r}$ for some K, μ since geodesics diverge exponentially fast:

$$Ke^{\mu r} \leq \text{length of green path} \leq (6\lambda + 2)r + C$$

Since exponential functions grow faster than linear functions,

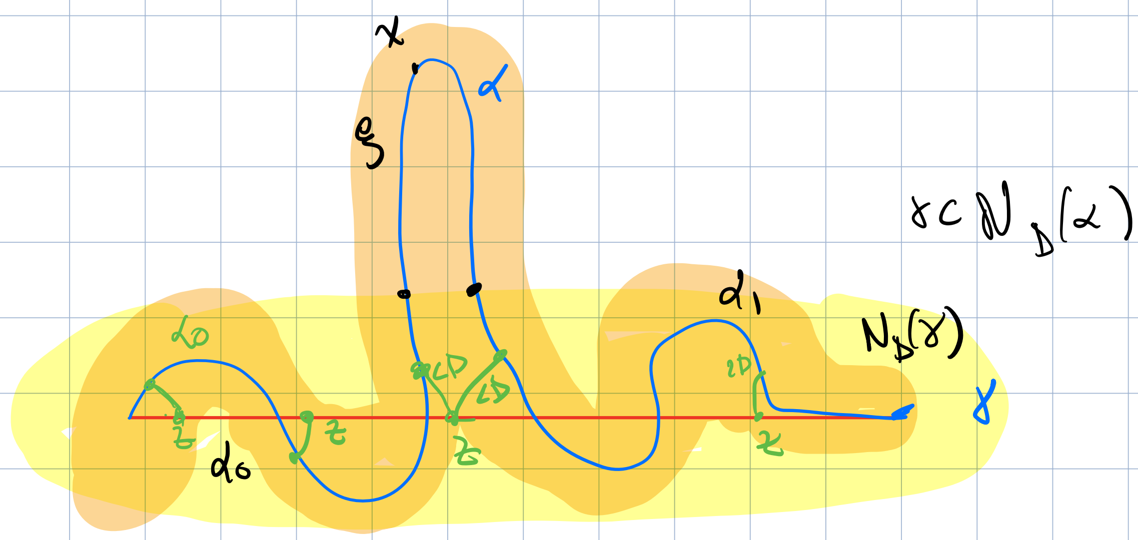
this means $r \leq D$ for some constant D. (which depends on

K, μ, C – and K, μ depend on λ, C, δ)

✓

Still have to show $\alpha \subset N_{D'}(\gamma)$! ie, given $x \in \alpha$, there is $z \in \gamma$ s.t. $d(x,z) < D' = D'(\lambda, c, \delta)$

<u>Proof</u> Suppose not. Let u, v be endpoints of an interval ξ on α that leaves $N_D(\gamma)$:



Decompose $\alpha = \alpha_0 \cup \xi \cup \alpha_1$

Let $z \in \gamma$. z is not close to anything in ξ but by the first part it is close to <u>some</u> point in α, so it is close to α_0 or α_1 (or both!)

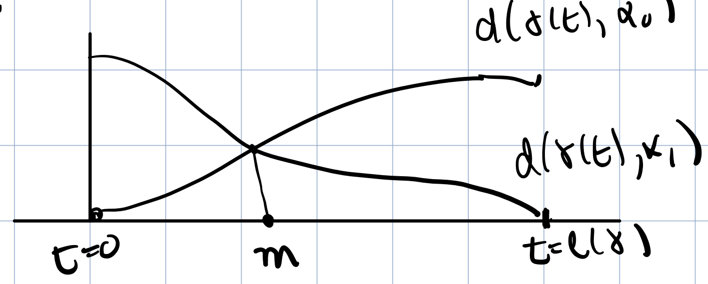
It starts close to α_0 on the left, then is close to α_1 on the right. Since α and γ are continuous, it is close to <u>both</u> at some z in between:

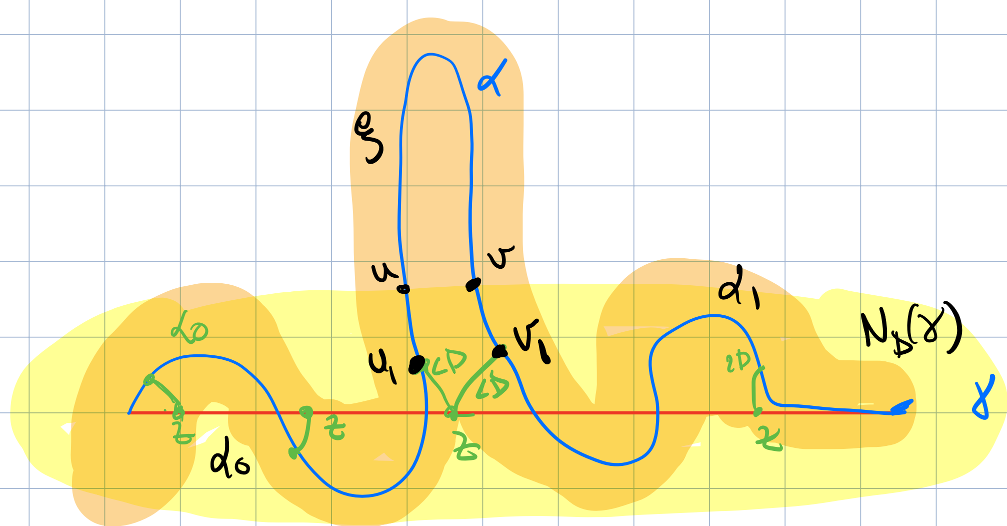
proof: α_0, γ continuous $\Rightarrow d(\gamma(t), \alpha_0)$ continuous

α_1, γ continuous $\Rightarrow d(\gamma(t), \alpha_1)$ continuous

$t=0 \Rightarrow d(\gamma(t),\alpha_0)=0$ $\quad t=\ell(\gamma) \Rightarrow d(\gamma(t),\alpha_1)=0$

So the graphs of $t \mapsto d(\gamma(t), \alpha_i)$ cross at some point $t=m$.





Let $z=\gamma(m)$, $u_1=\alpha(s)\in\alpha_1$, $u_2=\alpha(t)\in\alpha_2$

with $d(z,u_1)=d(z,u_2)\le D$

$\ell(\alpha|_{[s,t]}) \le \lambda d_X(\alpha(s),\alpha(t))+C \le \lambda 2D + C.$

$\Rightarrow$ every point of ξ is within $D' = D+\lambda D + C/2$

from a point of γ

ie $\alpha \subseteq N_{D'}(\gamma)$ ✓

(This theorem is what we needed to complete the proof that hyperbolicity is a quasi-isometry invariant.)

.

Hyperbolic groups

Since hyperbolicity is a quasi-isometry invariant we can define hyperbolic groups:

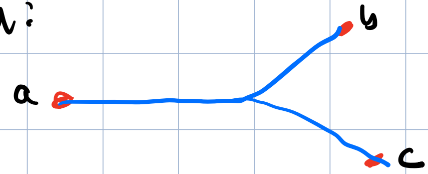
<u>Definition</u> A finitely-generated group G is **hyperbolic** if it has a hyperbolic Cayley graph

<u>Examples</u>

⓪ Finite groups - $\mathcal{C}(G,S)$ has finite diameter

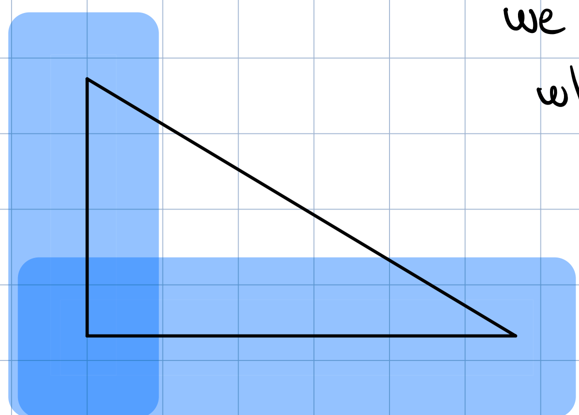
① $F(S)$

$\mathcal{C}(F(S),S)$ is a <u>tree</u>. In a tree, triangles are <u>very</u> thin:



in fact trees are 0-hyperbolic!

② $\mathbb{Z}^2$ is <u>not</u> hyperbolic: For any δ, we can find a triangle T whose third side is not contained in a δ-neighborhood of the other 2 sides,



④ $\pi_1(S_g)$, $g \geq 2$ acts properly and cocompactly on $\mathbb{D}$, so is quasi-isometric to $\mathbb{D}$, which has thin triangles so is (Gromov) hyperbolic

Since $\mathcal{C}(\pi_1 S_g) \sim_{qi} \pi_1(S_g) \sim_{qi} \mathbb{D}$,

$\mathcal{C}(\pi_1 S_g)$ is also hyperbolic ✓

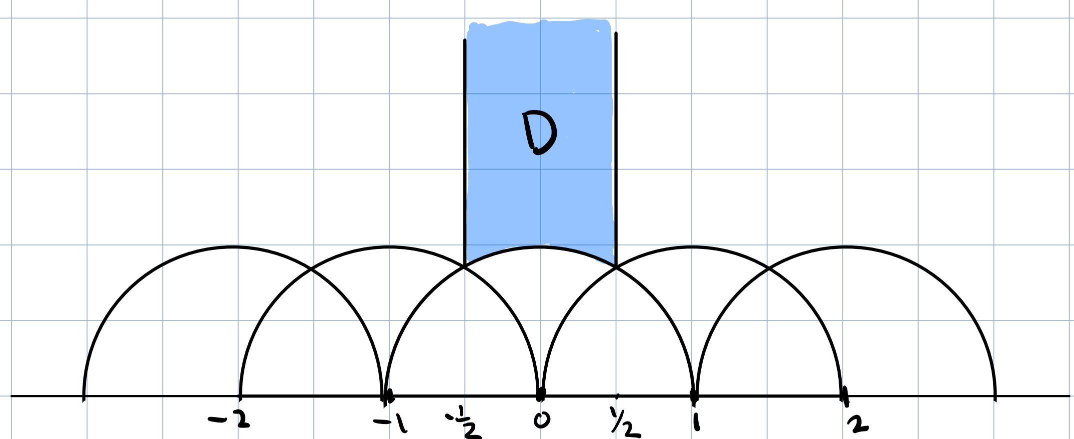
⑤ $SL(2,\mathbb{Z})$ acts on $\mathbb{H}$ properly, but <u>not</u> cocompactly, so can't use that to decide whether $SL(2,\mathbb{Z})$ is hyperbolic

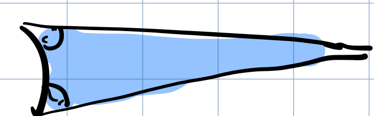
Let $D = \{ z \mid |z| > 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \}$

Claim: Translates of D cover $\mathbb{H}$,

ie if $w \in \mathbb{H}$, then $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $z \in D$.



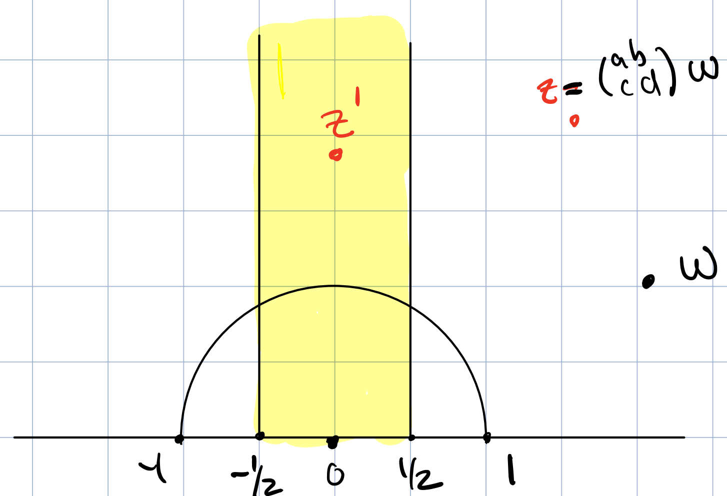
(The quotient $SL_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{stab}(D) \backslash D =$ 
is called the <u>modular</u> <u>curve</u>
by algebraic geometers and number theorists)

Proof

Claim For any $w \in \mathbb{H}$ there is some $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} w$ with $\operatorname{Im}(z)$ maximal

Pf $\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = \dfrac{aw+b}{cw+d}$ has imaginary part $\dfrac{\operatorname{Im}(w)}{|cw+d|^2}$. This is $\geq \operatorname{Im}(w)$ iff $|cw+d|^2 \leq 1$, ie $(cx+d)^2 + (cy)^2 \leq 1$. There are only finitely many integer solutions to this inequality, so $\operatorname{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix} w$ has a maximum $m(w)$

② Now let $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} w$ with $\operatorname{Im}(z)$ maximal



Translate z to z' in the strip $-\frac{1}{2} \leq \operatorname{Re}(z) \leq \frac{1}{2}$ by a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

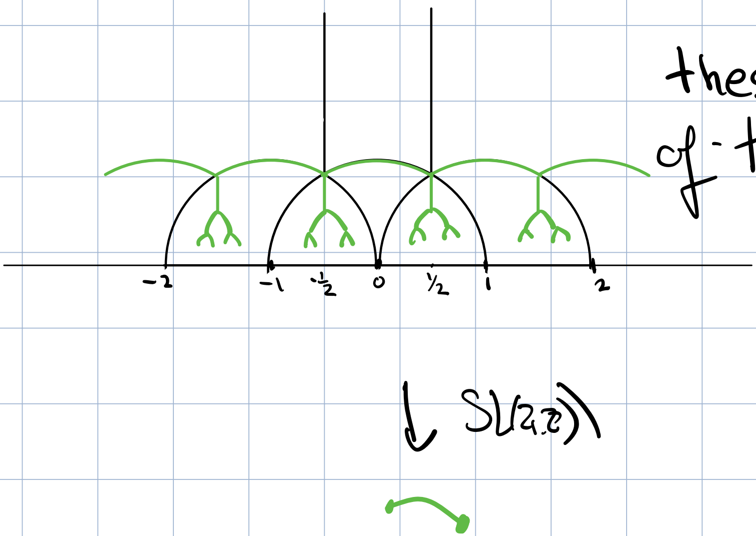
Claim: $|z'| \geq 1$. If not, then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z' = -\dfrac{1}{z'}$ has imaginary part

$$\frac{\operatorname{Im}(z')}{|z'|} = \frac{\operatorname{Im}(z)}{|z'|} > \operatorname{Im}(z),$$ contradicting maximality of $\operatorname{Im}(z)$. ✓

Now inside $\mathbb{H}$ we can find a tree T that is invariant under the action of $SL(2,\mathbb{Z})$ and has compact quotient, namely

$T = \{ w \mid m(w) = 1 \}$, ie

these are the images of of the bottom boundary of D.



So $SL_2(\mathbb{Z}) \underset{qi}{\sim} T$,

and $SL_2(\mathbb{Z})$ <u>is</u> hyperbolic!

Properties of hyperbolic groups

How can you use the geometry of the Cayley graph to prove algebraic facts about G?

<u>Theorem</u> If G is hyperbolic, then it has a finite presentation.

<u>Idea</u>: Let S be a finite generating set for G. We want to find $R \subseteq F(S)$ such that any word w in the generators that evaluates to the identity e_G is a product of conjugates of elements of R.

Since $w = e_g$, it gives a loop in $\mathcal{C} = \mathcal{C}(G,S)$.

Key: In a hyperbolic space, long loops have short segments that are not geodesics, where "short" depends only on δ.

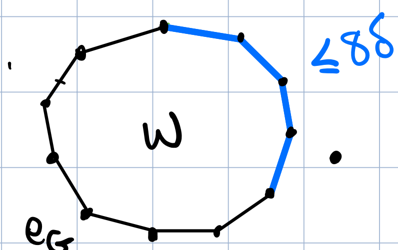
Not true if $\mathcal{C}(G,S)$ is not hyperbolic,

eg Look at $\mathcal{C}(\mathbb{Z}^2, (1,0), (0,1))$. In a $k \times k$ square, every segment of length $\leq k$ is a shortest path, ie a geodesic.

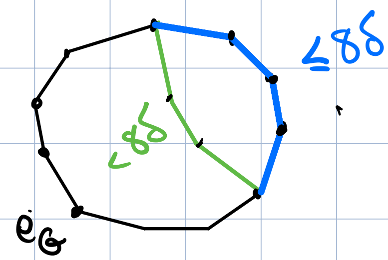
Here's a precise statement.

<u>Proposition</u> If w is any loop in a δ-hyperbolic Cayley graph $\mathcal{G}$, it has a segment of length $\leq 8\delta$ that is not a geodesic.

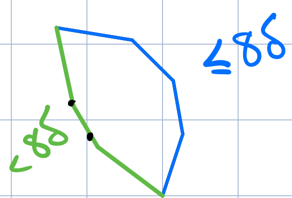
The theorem follows: Let R be the set of all words in S that evaluate to e_G and have length $< 16\delta$. (This is a finite set!)
Then w contains more than half of a relator:
If $\ell(w) < 16\delta$ it <u>is</u> a relater. Otherwise



. Draw a geodesic

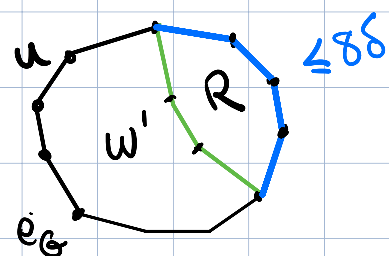


Then



has length $< 16\delta$, so is in R.

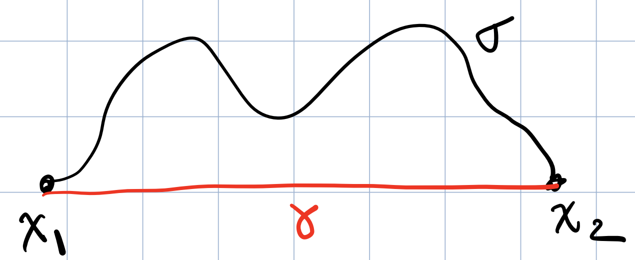
Then $w = (uRu^{-1})w'$:



with w' shorter than w, and we can continue until w is a product of conjugates of relators.

To prove the proposition:

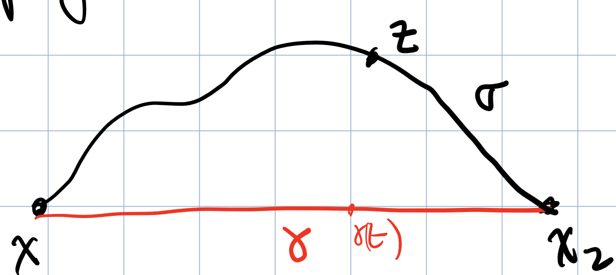
Suppose all segments of a path σ that have length $\leq 8\delta$ are geodesics.
Let γ be a geodesic between the endpoints x_1 and x_2 of σ:



<u>Lemma</u> $\sigma \subseteq N_{6\delta}(\gamma)$

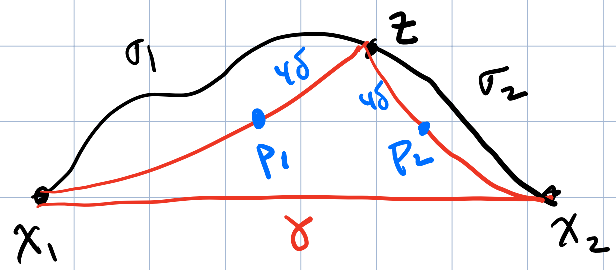
(so if σ is the loop ω, it is contained in $B_{6\delta}(e_G)$
But if ω has length $\geq 16\delta$ and all segments of length $\leq 8\delta$ are geodesics, then ω leaves $B_{6\delta}(e_G)$!)

<u>Proof of lemma.</u> Let z be a point on σ.



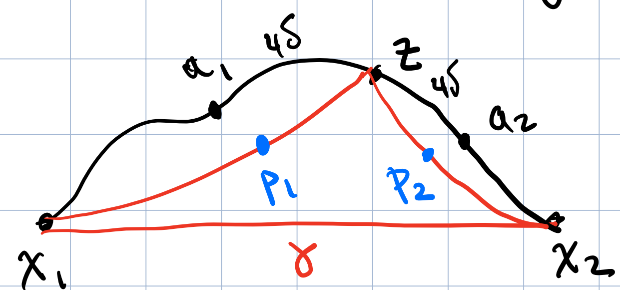
If σ is a geodesic, then $d(z,\gamma) < \delta$, so we are done.
(consider the triangle with vertices x_1, x_2, x_2)
In fact.
If $d(z, x_2) = d(\gamma(t), x_2)$, then $d(z, \gamma(t)) < 2\delta$.

If σ is not a geodesic, draw geodesics $[x_j, z]$ and mark points p_1, p_2 on them at distance 4δ from z:



z cuts σ into two pieces, σ_1 from x to z and σ_2 from z to y

Mark points a_i on σ_i at distance 4δ from z along σ (so the segment a_1 to a_2 is geodesic.)



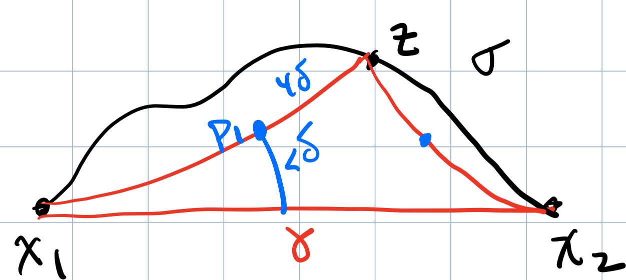
<u>Claim</u> $d(p_i, a_i) < 2\delta$.

Then $d(a_1,p_1) + d(p_1,p_2) + d(p_2,a_2) \geq d(a_1,a_2)$

so $d(p_1,p_2) \geq d(a_1,a_2) - d(a_1,p_1) - d(a_2,p_2)$

$$\geq 8\delta - 2\delta - 2\delta = 4\delta$$

$\therefore$ $d(p_1,p_2) > 2\delta$, so $d(p_1,\gamma) < \delta$, so $d(z,\gamma) < 5\delta$ and we are done.

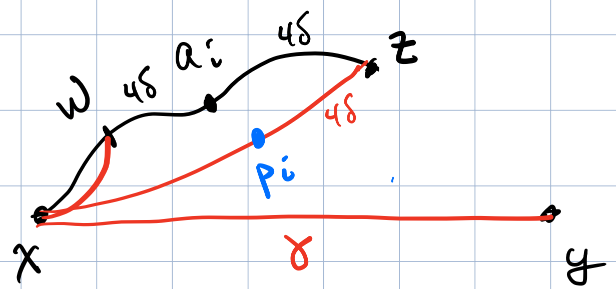


To prove the claim, we'll induct on the length of the σ_i

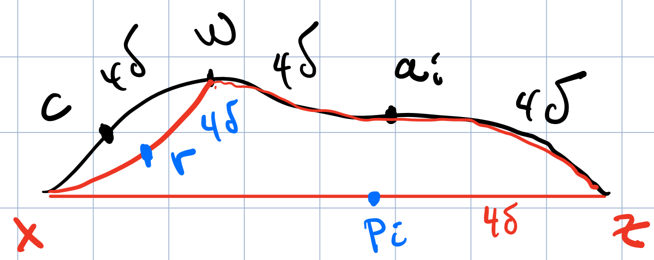
If σ_i has length $\leq 8\delta$ it is geodesic

so $\quad d(p_i, a_i) < 2\delta$

If σ_i is not a geodesic, cut off another 4δ-piece at a point w:



This is the same picture we had before, but with shorter pieces (and σ from w to z is already geodesic):



so by induction $d(r, c) < 2\delta$

$$\text{So} \quad d(a_i, r) \geq d(a_i, c) - d(r, c)$$
$$\geq 8\delta - 2\delta = 6\delta$$

So $d(a_i, [x,z]) < \delta$, and $d(a_i, p_i) < 2\delta$. ∎

The proposition we just proved can also be used to show that a hyperbolic group has only finitely many conjugacy classes of finite-order elements (See Exercise sheet)

Dehn functions

Suppose G is hyperbolic, and let $G = \langle S \mid R \rangle$ be a presentation

If $w \in F(S)$ with $w = 1$ in G, then $w = \prod_{i=1}^{k} h r_i h^{-1}$ for $r_i \in R$.

The Dehn function of $\langle S \mid R \rangle$ measures the size of k as a function of $|w|$, ie

<u>Def</u>: The Dehn function $d: \mathbb{Z} \to \mathbb{Z}$ is

$$d(\ell) = \max_{|w| \leq \ell,\ w =_G e_G} \min \left\{ k \mid w = \prod_{i=1}^{k} h r_i h^{-1} \right\}$$

Also called the isoperimetric function (says how many 2-cells in the Cayley complex you need to fill in a loop of length ℓ in the 1-skeleton = Cayley graph)

For the presentation we constructed (R = words of length $< 16\delta$), we showed the total number of relators you need for a word of length ℓ is $\leq \ell$.

ie d is linear: $d = a\ell + b$, $a = 1$

It turns out: the Dehn function for *any* presentation of a hyperbolic group is linear, and in fact this characterizes hyperbolic groups:

Theorem G is hyperbolic if and only if it has a linear Dehn function

Subgroups of hyperbolic groups

Theorem A If G is hyperbolic, it cannot contain a copy of $\mathbb{Z}^2$

Theorem B If G is hyperbolic and infinite, it has an infinite order element.

Theorem C If G is hyperbolic with infinitely many ends, it contains a copy of F_2.

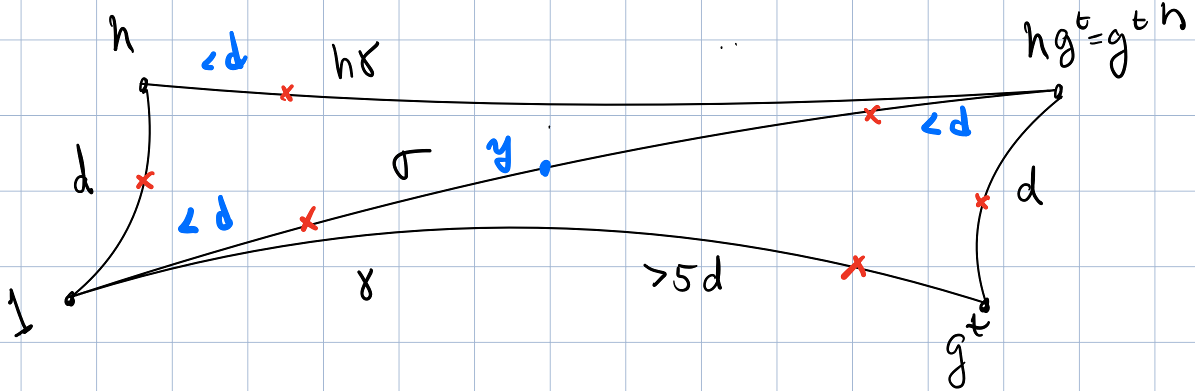
Let G be a hyperbolic group. Theorem A follows from

Proposition If $g \in G$ has infinite order, then $\langle g \rangle$ has finite index in its centralizer $C\langle g \rangle$

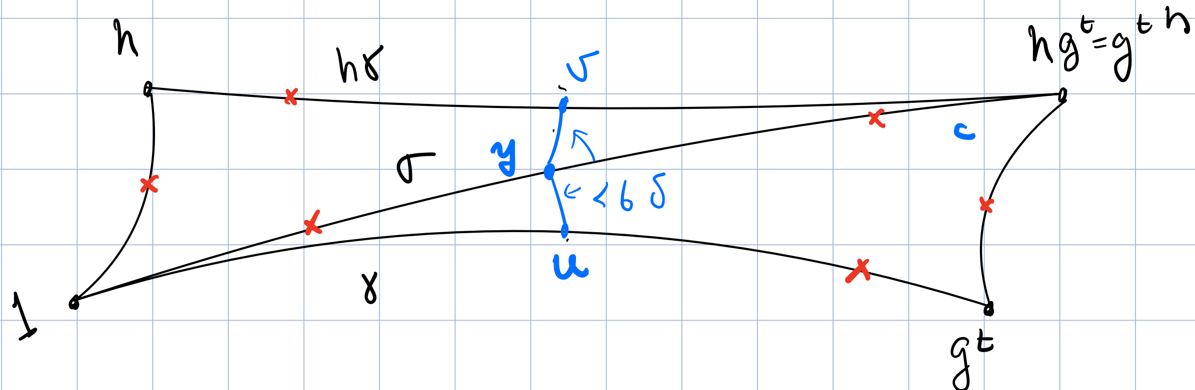
Proof Let $h \in C(g)$, ie $hg = gh$

Choose t large enough so that $d(1, g^t) > 5 \cdot d(1, h)$

Let γ be a geodesic 1 to g^t,
σ a geodesic 1 to $g^t h = h g^t$,
y = midpoint of σ.

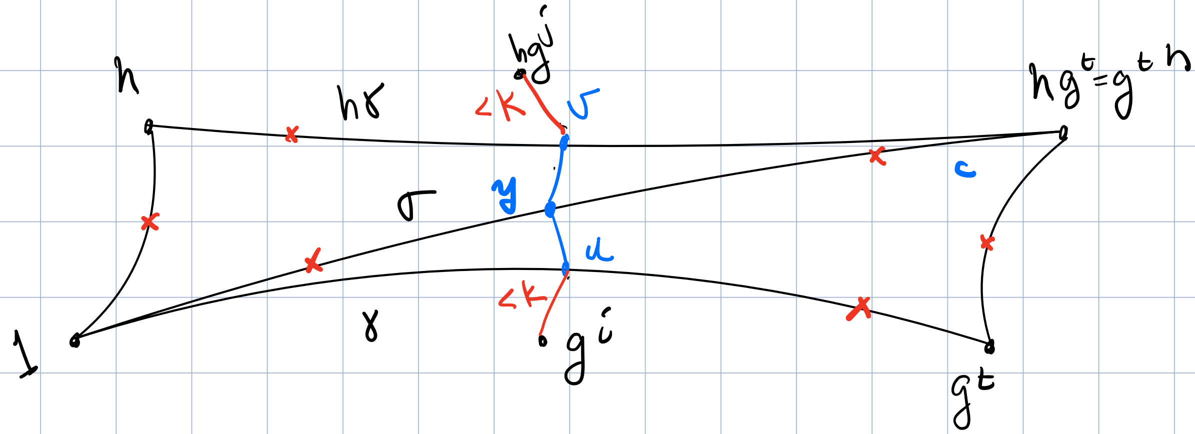


Since $\ell(\sigma) > 4d$, y is not close to the interior points of either triangle,
so there are $u \in \gamma$, $v \in h\gamma$ s.t. $d(u,y)$, $d(v,y) < 6\delta$, so $d(u,v) < 12\delta$



(*) Suppose we know $i \longrightarrow g^i$ is a quasi-isometric imbedding.

Then we proved $\{g^i \mid 0 \le i \le t\}$ stays a bounded distance from γ, ie $\exists K, i, j$
s.t. $d(g^i, u) < K$
$d(hg^j, v) < K$



So $d(g^i, hg^j) < 2K + 12\delta$

$$\parallel$$

$$d(1, hg^{j-i}) < 2K + 12\delta$$

So the coset $h\langle g\rangle$ enters $B_{2K+12\delta}(1)$

But $B_{2K+12\delta}(1)$ is a finite ball, and the cosets of $\langle g\rangle$ in $C\langle g\rangle$ are disjoint, so only finitely many of them can intersect $B_{2K+12\delta}$

We just proved they <u>all</u> do!
So there are only finitely many of them ✓

<u>Corollary</u>: If G is hyperbolic, it does not contain $\mathbb{Z}^2$.

<u>Proof</u>: the centralizer of any $g \in \mathbb{Z}^2$ is at least all of $\mathbb{Z}^2$, and $\langle g\rangle$ doesn't have finite index in $\mathbb{Z}^2$.

We still need to prove $(*)$

Proposition Suppose $g \in G$ has infinite order. Then the map $\mathbb{N} \longrightarrow G$ sending $n \longmapsto g^n$ is a quasi-isometric embedding.

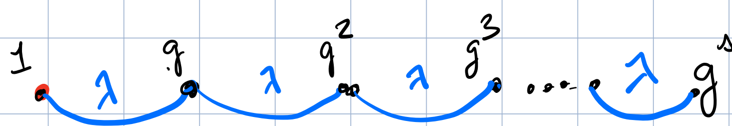
Proof: Since g has infinite order, $\{g^i\}$ leaves every ball around 1 in the Cayley graph $\mathcal{G}$.

We need to find λ, C s.t.

$$\frac{s}{\lambda} - C \leq d_{\mathcal{G}}(1, g^s) \leq \lambda s + C$$

The RH inequality is just the triangle inequality:

Let $\lambda = d(1, g)$



Then $d(1, g^s) \leq \lambda s$ ✓

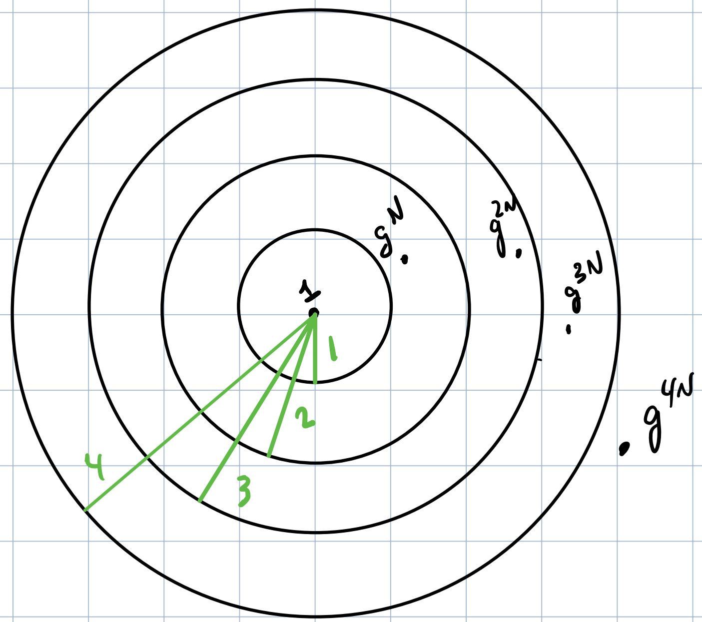
The hard part is the LH inequality: $d(1, g^s) \geq \frac{1}{\lambda} s - C$

We will show

(*) There is N such that for any $k > 0$,

$$d(1, g^{Nk}) \geq k.$$

(in fact $N = 3 \cdot \#\{i \mid g^i \in B_{12\delta}(1)\}$ works)



Idea: $d(1,g^N) \geq 1$

$d(1,g^{2N}) \geq 2$

etc.

$d(1,g^{kN}) \geq k$

Then for $Nk < s < (N+1)k$

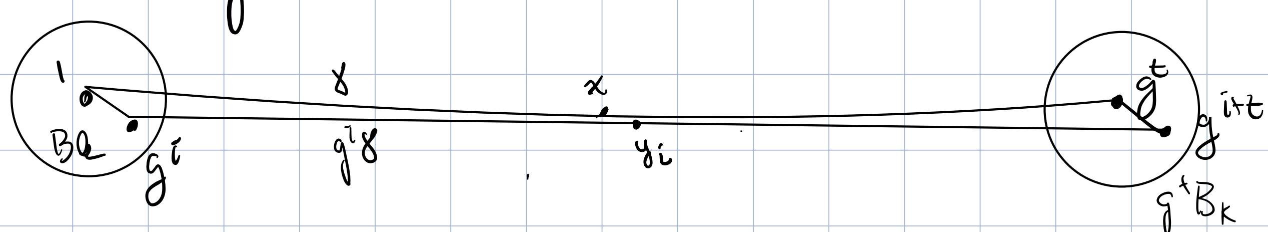
g^s can only go a bounded distance back into B_k.

ie For any s, write $s = kN + j$ with $j < N$ and let $m = \max_{j<N} d(1,g^j)$

$$\begin{aligned}
\text{Then } d(1,g^s) &\geq d(1,g^{kN}) - d(g^{kN}, g^{kN+j}) \\
&= d(1,g^{kN}) - d(1,g^j) \\
&\geq k - m \\
&= \frac{(s-j)}{N} - m \geq \frac{s}{N} - (1+m) \checkmark
\end{aligned}$$

To prove $(*)$ we will first bound how many g^i are in B_k for a given k.

Intuition: Suppose $g^i \in B_k$. For t large, a geodesic quadrilateral with vertices $1, g^i, g^t, g^{t+i}$ is very thin at the center



so you *expect* the top and bottom midpoints to be at most k apart, ie $d(x, y_i) \leq k$

Since G acts freely on $\mathcal{G}$, midpoints of the geodesics $g^i \gamma$ are all distinct. Since they all lie in $B_{2k}(x)$ there are at most $2k+1$ of them

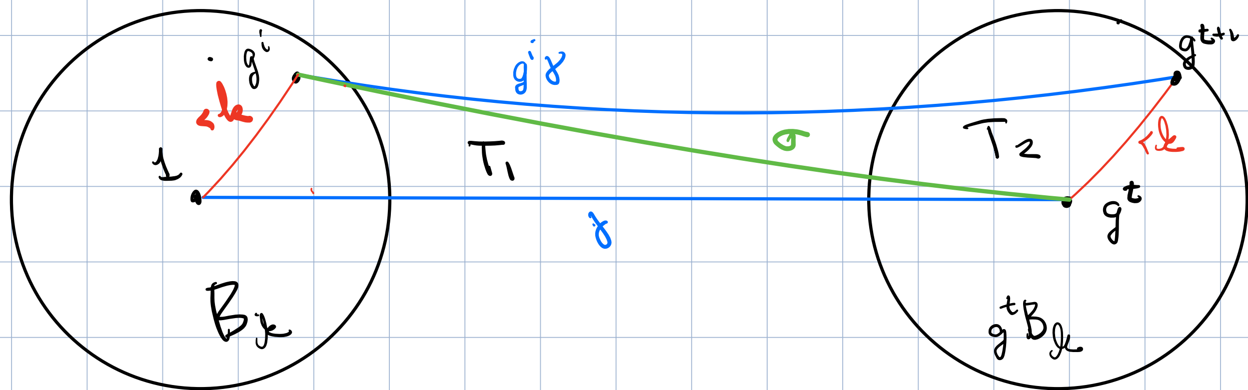
ie $\#\{i \mid g^i \in B_{2k}\} < 2k+1$.

But your count may be off because of δ ...

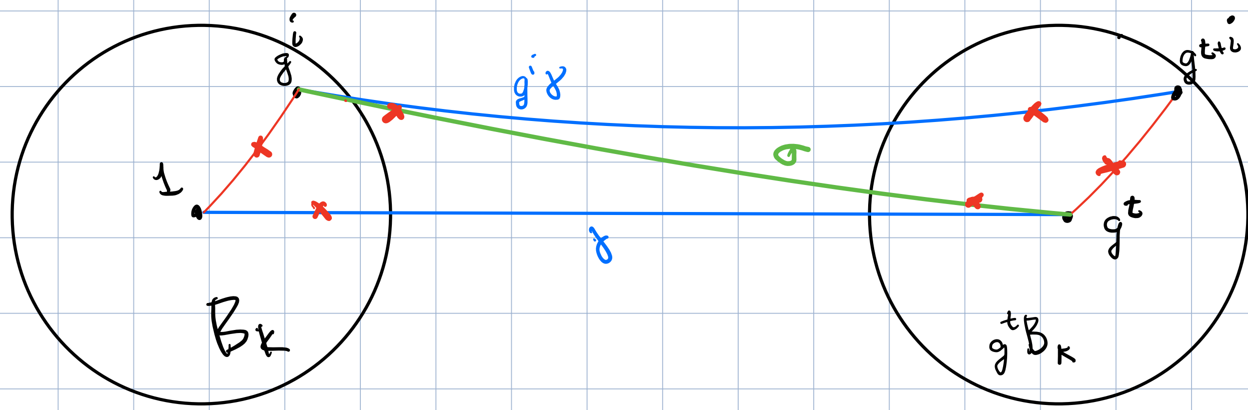
Here is a formal argument:

Fix k, and choose t so that $d(1, g^t) > 8k + 12\delta$

Draw geodesics γ from 1 to g^i, σ from g^i to g^t so that γ, σ, and $g^i \gamma$, together with geodesics $[1, g^i]$ and $[g^t, g^{t+i}]$ form two triangles T_1 and T_2:



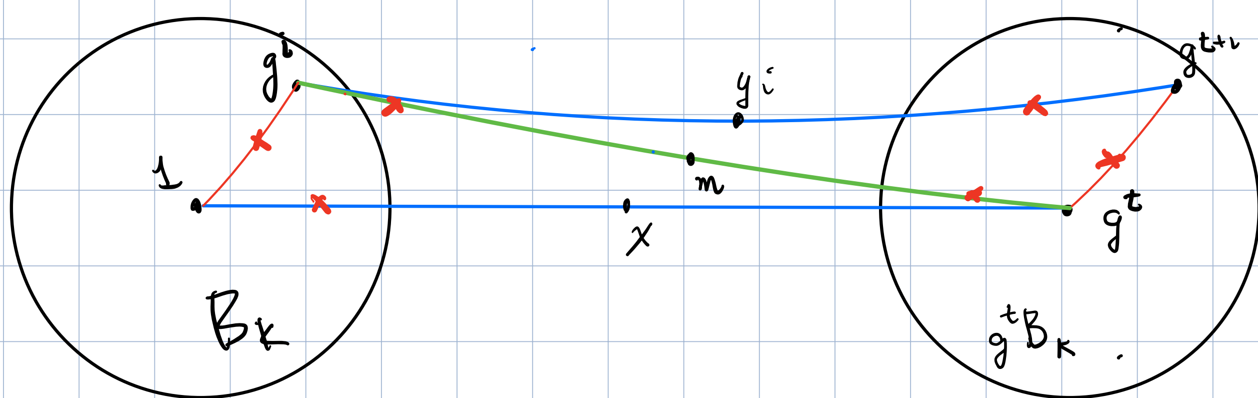
Because $d(1, g^t)$ is much larger than k, the incenters of T_1 are close to B_k:



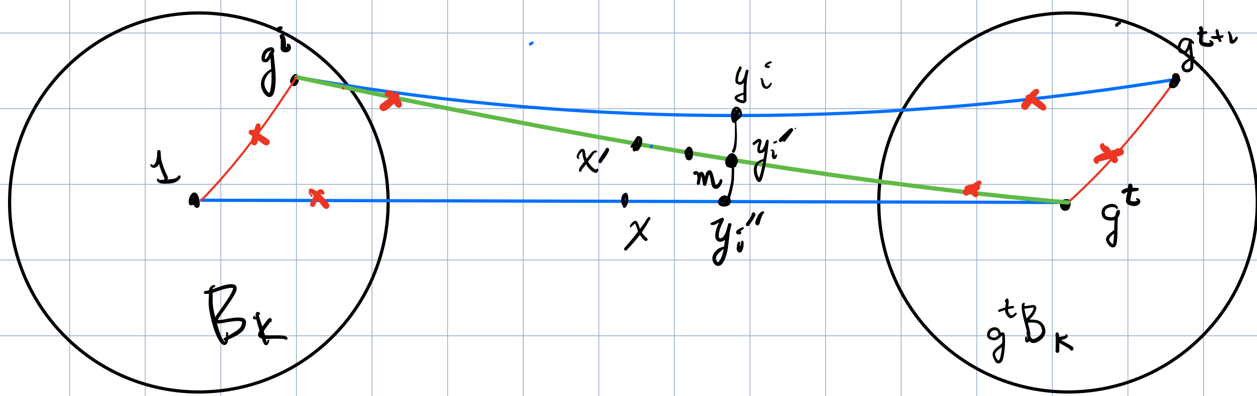
So points near the middle of γ and σ that are the same distance from g^t have distance $< 6\delta$.

Similarly, points near the middle of σ and $g^i\gamma$ that are the same distance from g^i are at most 6δ apart.

Let x = midpoint of γ, m = midpoint of σ and y_i = midpt of $g^i\gamma$

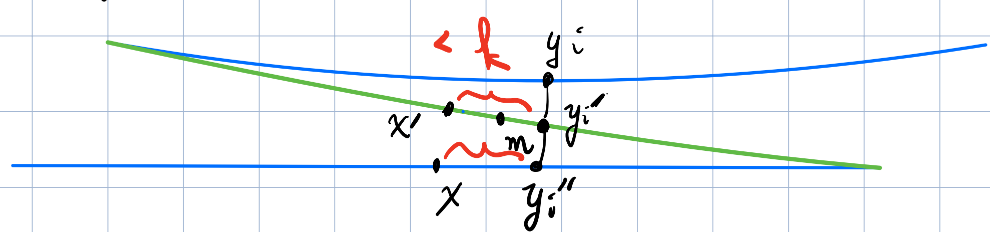


Let $y_i' \in \sigma$ be the point with $d(g^i, y_i') = d(g^i, y_i)$
$x' \in \sigma$ the point with $d(g^t, x') = d(g^t x)$
and $y_i'' \in \gamma$ the point with $d(g^t, y_i'') = d(g^t, y_i)$

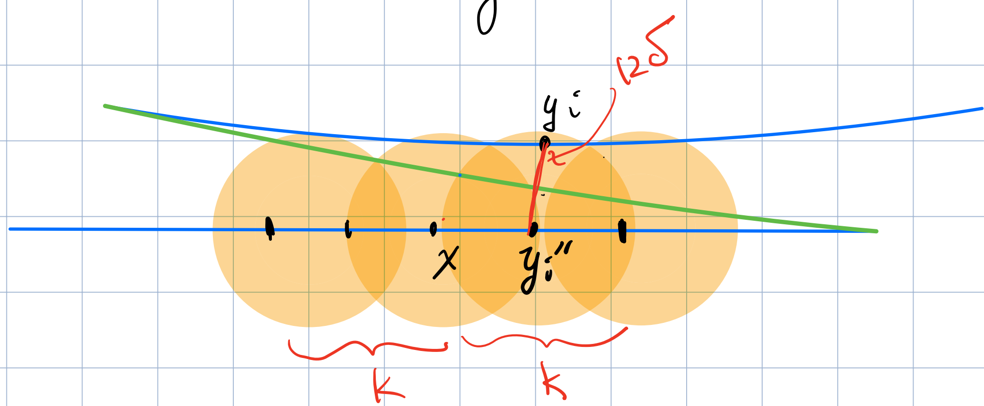


Then $d(y_i', m) = \frac{1}{2}\left(d(g^i, g^{t+i}) - d(g^i, g^i)\right)$
$\leq \frac{1}{2} k$

Similarly $d(x', m) \leq \frac{1}{2} k$, so $d(x, y_i'') < k$:



$d(y_i, y_i')$ and $d(y_i', y_i'')$ are both $< 6\delta$,
so $d(y_i, y_i'') < 12\delta$.
ie y_i is in the union of $2k+1$ balls of radius 12δ



Let $C = \#\{\text{vertices in } B_{12\delta}\}$
This says y_i is one of $(2k+1)\cdot C$ points.
Since the $y_i = g^i x$ are all distinct, there are at most $(2k+1)\cdot C$ elements g^i with $g^i \in B_k(1)$.

So for every k there is some number $e(k) \leq 3Ck$ st $d(1, g^{e(k)}) > k$ where $C = \#\{i \mid g^i \in B_{12\delta}\}$

Since $d(1,g^i) < i \cdot d(1,g)$, we also know $e(k) \geq k / d(1,g)$.

So $$\boxed{\frac{k}{d(1,g)} \leq e(k) \leq 3Ck}$$

<u>We claim</u>: for all k, $d(1, g^{3Ck}) \geq k$

<u>Proof</u> Suppose not, ie there is some k_0 with

$$d(1, g^{3Ck_0}) = k_0 - \varepsilon, \quad \text{with } \varepsilon \geq 1$$

For any k, write $e(k) = m \cdot 3Ck_0 + j$, with $j < 3Ck_0$

Then
$$\begin{aligned} d(1, g^{e(k)}) &\leq d(1, g^{m \cdot 3Ck_0}) + d(1, g^j) \\ &\leq m \cdot (k_0 - \varepsilon) + \max_{j < 3Ck_0} d(1, g^j) \\ &= m k_0 - m\varepsilon + C \end{aligned}$$

Since $e(k) > k/d(1,g)$ we can make $e(k)$ (and therefore m) arbitrarily large

So for k large enough, $mk_0 - m\varepsilon + C < mk_0$

so

$$d(1, g^{e(k)}) < mk_0 \leq e(k)/3C \leq 3Ck/3C = k,$$

contradicting the definition of $e(k)$.

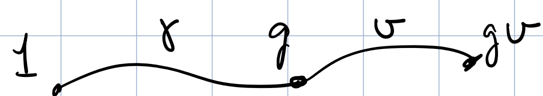
Infinite order elements of hyperbolic groups

Theorem B If a hyperbolic group G is infinite, it contains an element of infinite order

To prove this, we use the concept of conetype of an element in a Cayley graph.

Definition Let $\mathcal{C} = \mathcal{C}(G,S)$ be a Cayley graph for G, and $g \in G$. The cone type $c(g)$ is the set of words v such that
$$d(1, gv) = d(1,g) + \ell(v)$$

Ie if γ is a geodesic path from 1 to g, it's the set of paths starting at g such that the concatenation $\gamma \cdot v$ is a geodesic:

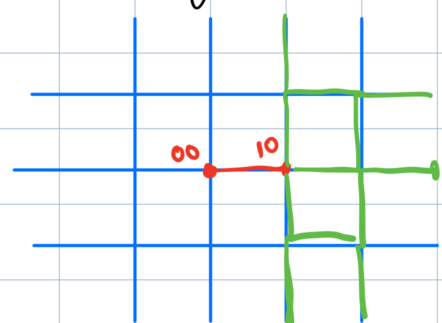


Examples

1 $\mathbb{Z}^2$, $(1,0), (0,1)$ has $\underline{9}$ cone types

$c(0,0)$ is $\underline{\text{all}}$ geodesic paths

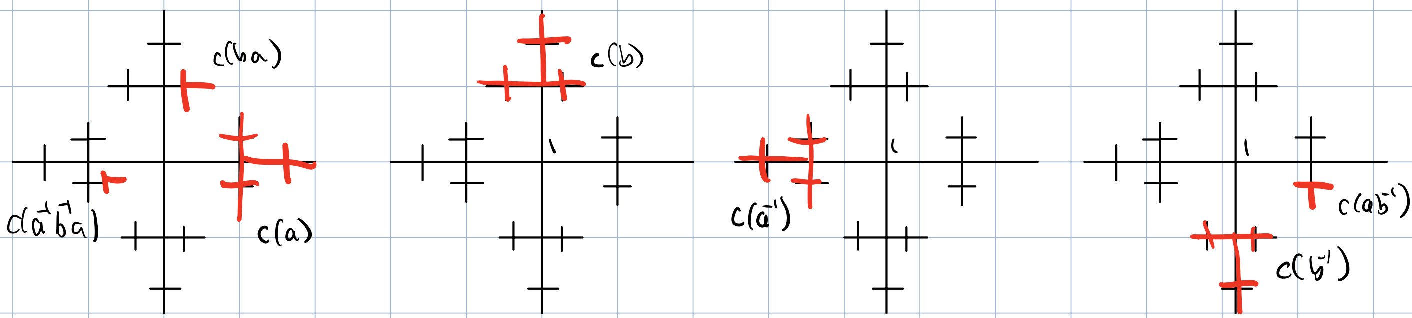
$c(1,0)$ is all geodesic paths from $(0,0)$ to (m,n) with $m \geq 0$



$c(1,1)$ is all geodesic paths in the first quadrant

Notice that isometries don't preserve cone type.

2. F_2 has 5 cone types: $c(1)$ = all geodesics, plus



Note $c(g)$ only depends on the last letter of g!!

<u>Proposition</u> A hyperbolic group has only finitely many cone types.

<u>Proof</u>
We claim that the cone type of g is determined by the "tail"

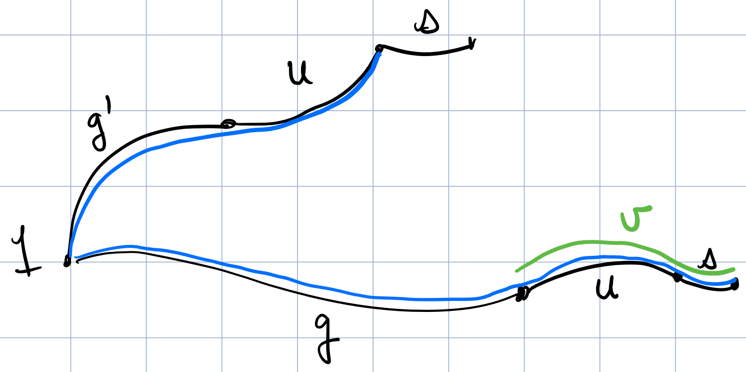
$$T(g) = \{u \in B_{6\delta+2} \mid d(1, gu) < d(1,g)\}$$

Since $B_{6\delta+2}$ is finite, this implies there are only finitely many possibilities for $T(g)$, so for $c(g)$

So suppose $T(g) = T(g')$. To show $c(g) = c(g')$ we will induct on the length $\ell(v)$

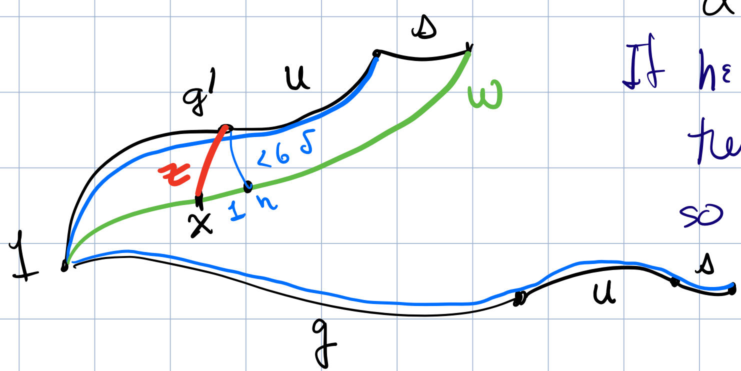
If $\ell(v) = 0$, this is trivial, so assume $\ell(v) \geq 1$

So assume $\ell(v) \geq 1$, and suppose $v \in c(g)$.
Write $v = us$ with $s \in S$, $\ell(u) = \ell(v) - 1$, so by induction, $u \in c(g')$



blue paths are geodesic

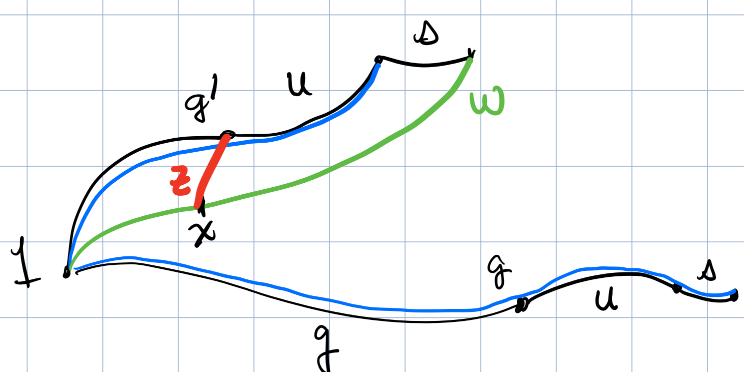
Suppose $g'us$ is not a geodesic. Draw a geodesic w from 1 to $g'us$, mark $x \in w$ with
$$d(1,x) = d(1,g') - 1$$



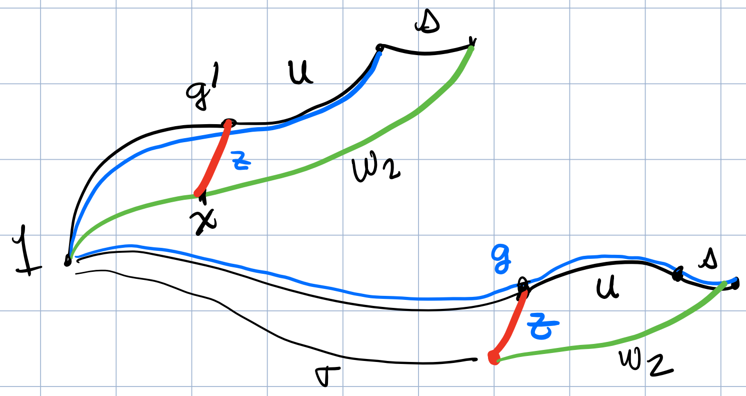
If $h \in w$ $d(1,h) = d(1,g')$, then $d(h,g') \leq 6\delta$

so $d(g',x) \leq 6\delta + 1 < 6\delta + 2$

So a geodesic z from g' to x is in $T(g')$



$$\ell(w) = d(1,x) + d(x,w) = d(1,g') - 1 + d(x,w) < d(1,g') + |u| + 1$$
$$\Rightarrow d(x,w) < \ell(u) + 2 \leq \ell(u) + 1$$



$T(g') = T(g) \Rightarrow z \in T(g) \Rightarrow d(1, gz) < d(1, g)$

Let σ be a geodesic from 1 to gz

Then the path $\sigma \circ w_2$ has length

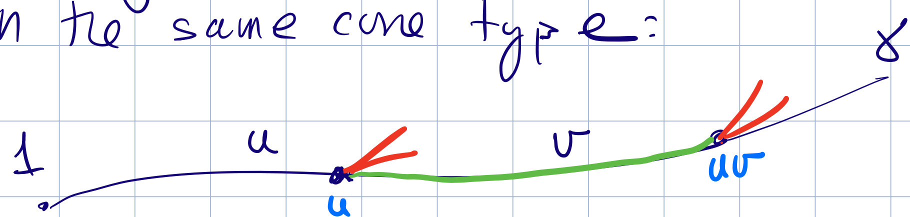
$< d(1,g) + \ell(u) + 1 = d(1, gus)$

contradicting the fact that the path gus is a geodesic

Corollary: If G is an infinite hyperbolic group, it has an element of infinite order.

Proof Since G is infinite and balls in $\mathcal{C}(G,S)$ are finite, we can find arbitrarily long geodesics starting at 1.

Choose one that is longer than the number of cone types; it must contain two vertices with the same cone type:



$d(1, u\sigma) = d(1,u) + \ell(\sigma)$

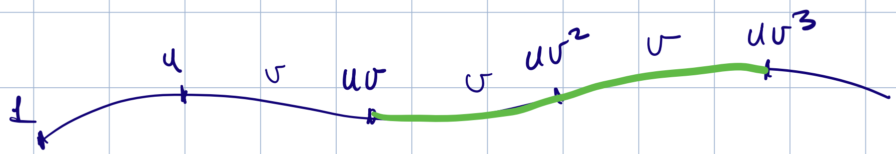
Since γ is a geodesic, σ is in the cone type of u

Since u and uv have the same cone type, v is also in the cone type of uv

so $d(1, uv^2) = d(1,u) + 2 \cdot \ell(v)$

so v^2 is in the cone type of u

So v^2 is also in the cone type of uv:



so $d(1, uv^3) = d(1,u) + 3\,\ell(v)$

etc: $d(1, uv^n) = d(1,u) + n\,\ell(v)$ for all n

Since $d(1, v^n) \geq d(1, uv^n) - d(1,u) \longrightarrow \infty$ as $n \to \infty$, v has infinite order.

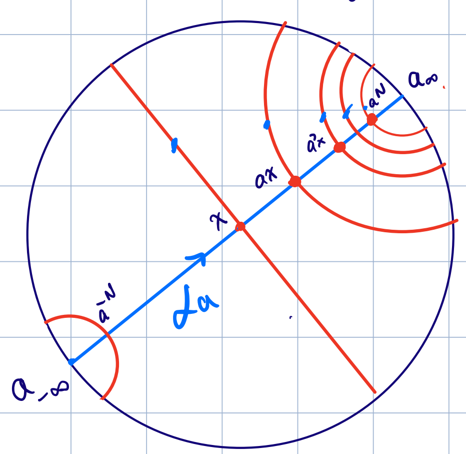
Free subgroups of hyperbolic groups

<u>Theorem C</u>: A hyperbolic group with infinitely many ends contains a copy of F_2

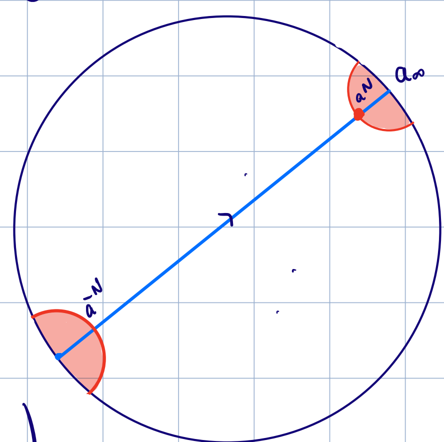
Idea: use $\mathcal{C}(G,S)$ as a ping-pong table, ie find $A, B \subseteq \mathcal{C}(G,S)$ such that $B \cap A \neq \emptyset$. and

$$a^n B \subset A$$
$$b^n A \subset B$$

Inspiration: elements $a \in SL(2,\mathbb{Z})$ with trace > 2 have an axis α_a, and a acts on $\mathbb{H}$ by translating along α_a (picture in $\mathbb{D}$)



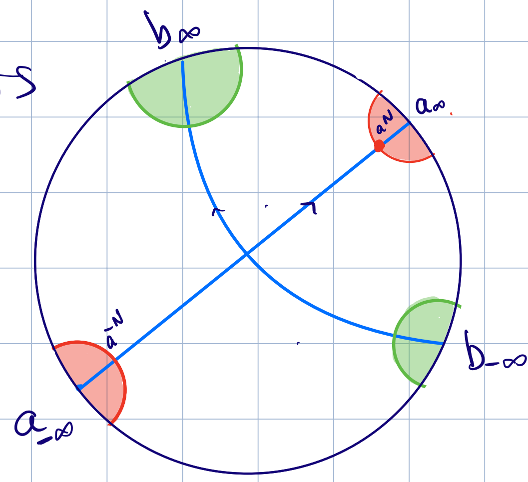
For $|m| > 2N$, everything in the unshaded area below is taken into the red shaded area by a^m:



a^+_∞ and a^-_∞ are "points at ∞", and the red regions are "N-neighborhoods" $V_N(a^+_\infty)$ and $V_N(a_{-\infty})$ of these points. Let $A = V_N(a_\infty) \cup V_N(a_{-\infty})$

If b has an axis with different endpoints b^+_∞, b^-_∞, let $B = V_N(b^+_\infty) \cup V_N(b^-_\infty)$
then for N large enough $A \cap B = \emptyset$
and for m large enough

$$\left.\begin{array}{l} a^{km} B \subset A \\ b^{km} A \subset B \end{array}\right\} \forall k \geq 1$$



so a^m and b^m generate a free group.

We want to show there is a similar picture for a hyperbolic group acting on its Cayley graph.

If G has infinitely many ends, it is infinite, so has an element g of infinite order

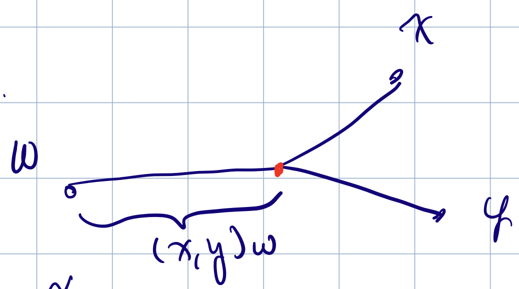
We know the map $\mathbb{N} \to G$ sending $i \longmapsto g^i$ is a quasi-isometric embedding, we want to define a "limit" g_∞ of this sequence, and a "neighborhood" $V_N(g_\infty)$

We use the Gromov product for this:

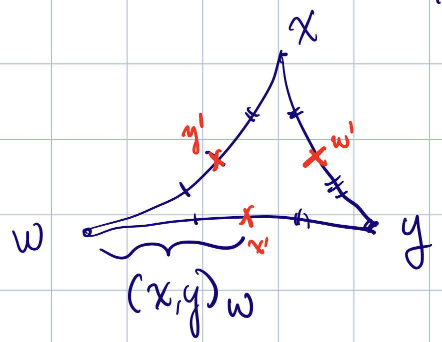
<u>Def</u> Let x, y, w be points in a metric space. The Gromov product $(x,y)_w$ is defined by

$$(x,y)_w = \frac{1}{2}\left[d(w,x) + d(w,y) - d(x,y)\right]$$

Example: In a tree.



In general

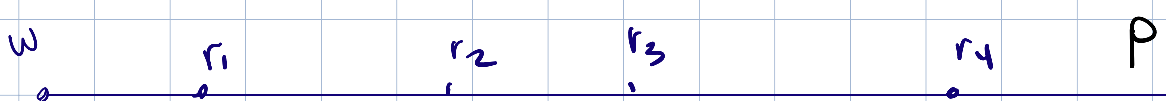


Red points are the interior points.

Def A sequence $\{x_i\}$ $i \in \mathbb{N}$ converges to ∞ (write $\{x_i\} \to \infty$) if

$$\boxed{\text{For every } R>0, \text{ there is } N \text{ such that } i,j > N \Rightarrow (x_i, x_j)_w > R.}$$

Example $\{r_i\}$ are points on an (infinite) geodesic ray in $\mathcal{G}$ starting at w with $d(w, r_i) \longrightarrow \infty$:

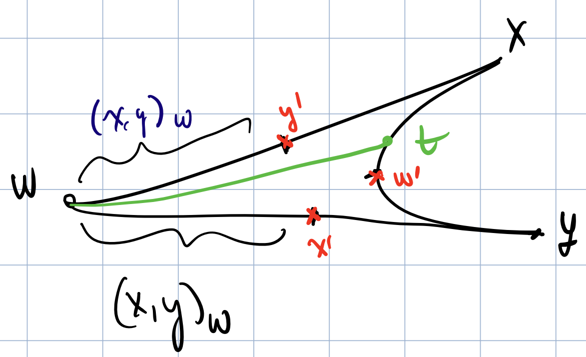


Then

$(r_i, r_j)_w = \min(d(w, p_i), d(w, p_j))$, so $\{r_i\} \longrightarrow \infty$

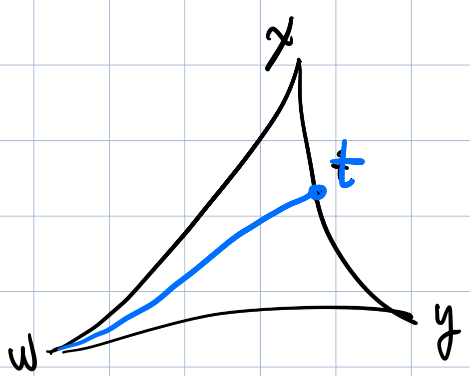
Does $(x,y)_w$ depend on the choice of w?

Lemma $(x,y)_w \le d(w,[x,y]) \le (x,y)_w + 2\delta$



Proof The second inequality is true because at least one of $d(x',w')$ or $d(y',w')$ is $\le 2\delta$.

For the first, choose $t \in [x,y]$ with $d(w,t) = d(w,[x,y])$

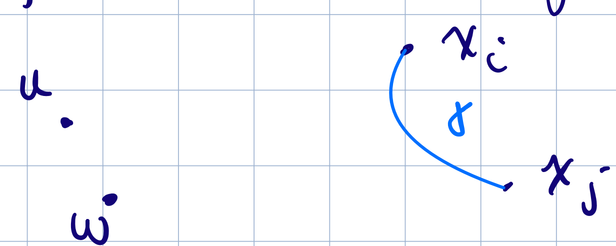


Then

$$\begin{aligned} d(w,t) &= \tfrac{1}{2}\left(d(w,t) + d(w,t)\right) \\ &\geq \tfrac{1}{2}\left(d(w,x) - d(x,t) + d(w,y) - d(y,t)\right) \\ &= \tfrac{1}{2}\left(d(w,x) + d(w,y) - d(x,y)\right) \\ &= (x,y)_w \quad \checkmark \end{aligned}$$

Corollary: $\{x_i\} \longrightarrow \infty$ is independent of the choice of w

Proof Suppose $u \neq w$. We want to compare $(x_i, x_j)_u$ and $(x_i, x_j)_w$. Let γ be a geodesic from x_i to x_j:

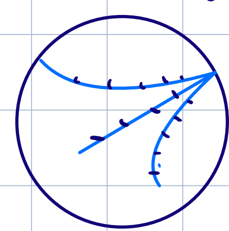


$$(x_i, x_j)_u \le d(u,\gamma) \le d(u,w) + d(w,\gamma)$$
$$\le d(u,w) + (x_i, x_j)_w + 2\delta$$

So if $(x_i, x_j)_u \longrightarrow \infty$, $(x_i, x_j)_w \longrightarrow \infty$ too (and conversely).

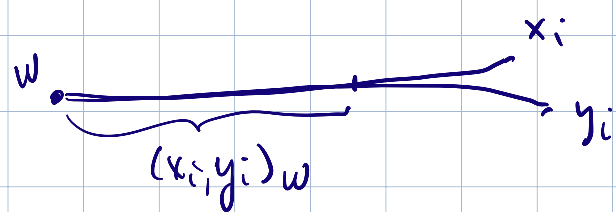
Now I want to define a "point at ∞" to be a sequence $\{x_i\} \longrightarrow \infty$

But different sequences can converge to the same point.



So define $\{x_i\} \sim \{y_i\}$ if

$$(x_i, y_i)_w \to \infty \quad \text{as } i \to \infty$$



Want to say this is an equivalence relation, and define a point at ∞ to be an equivalence class.

<u>**But**</u> If X is not hyperbolic, this may not be an equivalence relation (example in the Exercises)

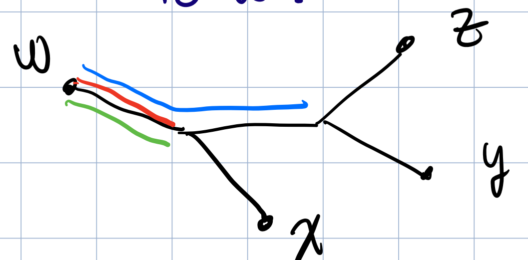
Proposition If X is hyperbolic, then $\sim$ is an equivalence relation

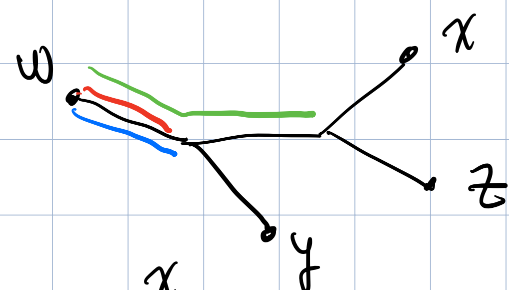
Reflexive and symmetric are obvious. To prove the proposition, we just have to verify transitivity

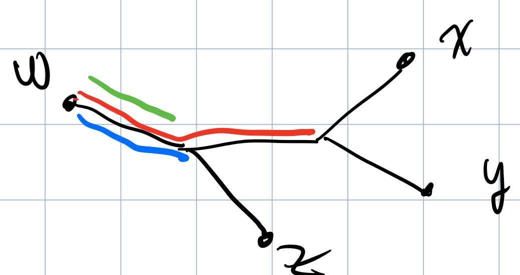
i.e. Suppose $\{x_i\} \to \infty$, $\{y_i\} \to \infty$, $\{z_i\} \to \infty$,
$\{x_i\} \sim \{y_i\}$ and $\{y_i\} \sim \{z_i\}$

To prove $\{x_i\} \sim \{z_i\}$ we need to understand the relation between $(x_i, y_i)_w$, $(y_i, z_i)_w$ and $(x_i, z_i)_w$

Intuition: In a tree, there are 3 cases, depending on which of x, y, or z is closest to w:

① 

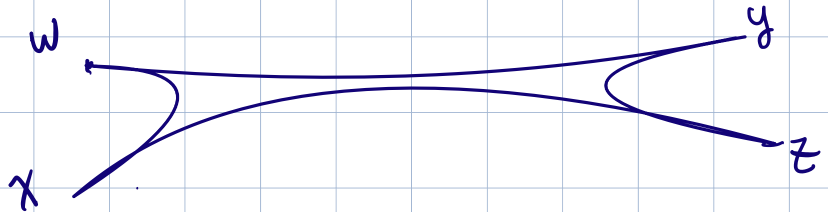
② 

③ 

$(x,y)_w$ $\quad$ $(y,z)_w$ $\quad$ $(x,z)_w$

in all cases $\boxed{(x,z)_w \geq \min\{(x,y)_w, (y,z)_w\}}$

In a hyperbolic metric space, the picture is just a slightly fatter version of this one:

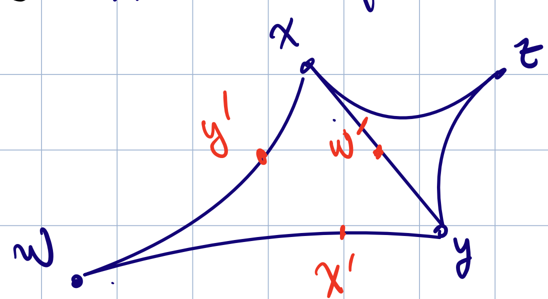


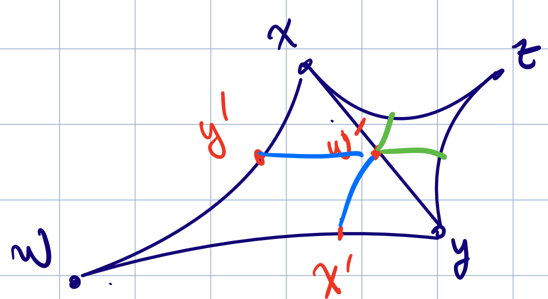
Lemma In a δ-hyperbolic metric space X

$$(x.y)_w \geq \min\left((x.z)_w, (z.y)_w\right) - 3\delta$$

for any $x, y, z, w \in X$.

Proof: Consider geodesic triangles with vertices w, x, y and x, y, z, and let w', y', z' be the internal points of $\Delta(wxy)$:





One of $d(y', w')$, $d(x', w')$ is $< 2\delta$

One of $d(w', [x,z])$, $d(w', [y,z])$ is $< \delta$

So $(x,y)_w + 3\delta \geq \min\left(d(w,[x,z]), d(w,[y,z])\right)$

$\geq \min\left((x,z)_w, (y,z)_w\right)$ $\square$

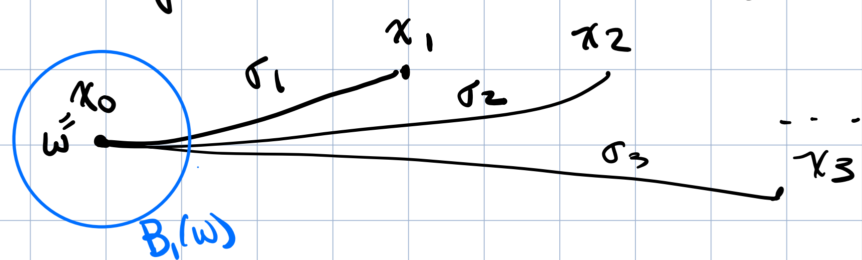
Proof of proposition Since $(x,z)_\omega \to \infty$ and $(y,z)_\omega \to \infty$, the lemma shows $(x,y)_\omega \to \infty$, ie $\{x_i\} \sim \{y_i\}$ $\square$

So now our points at infinity are well-defined. We'll call them boundary points $x_\infty \in \partial G$

We next claim that $\{g^i\} \to \infty$. We know $i \longmapsto g^i$ is a quasi-isometric embedding.

Proposition Suppose $\{x_i\}$ are the vertices of a (λ, C)-quasi-isometric embedding $\mathbb{N} \longrightarrow X$. Then there is an infinite geodesic ray p and a constant $K = K(\lambda, C, \delta)$ such that $\{x_i\} \subset N_K(p)$ and $p \subset N_K(\{x_i\})$.

Proof Choose a geodesic σ_i from $\omega = x_0$ to x_i, for each i:



An infinite number of σ_i agree on $B_1(\omega)$, since $B_1(\omega)$ is finite. Let p_1 be the first edge of any of these σ_i.
An infinite number of the σ_i that contain p_1 agree on $B_2(\omega)$
Let p_2 be the first two edges of any of these σ_i

continue passing to subsequences of the σ_i to define

$$p_1 \subset p_2 \subset p_3 \subset \dots$$

and let $p = \bigcup p_i$. This is an infinite geodesic ray.
We know quasi-geodesics stay close to geodesics, ie
There is a constant K depending only on δ, λ and C
such that

$$j \geq i \Rightarrow \{x_i\} \subseteq N_K(\sigma_j)$$
$$\text{and } \sigma_j \subseteq N_K(\{x_i \mid i \leq j\})$$

This implies ① $\{x_i\} \subset N_K(p)$ and ② $p \subset N_K(\{x_i\})$:

① Let $N > d(1, x_i) + K$ and choose $j > i$ such that $\sigma_j \supseteq p(N)$:

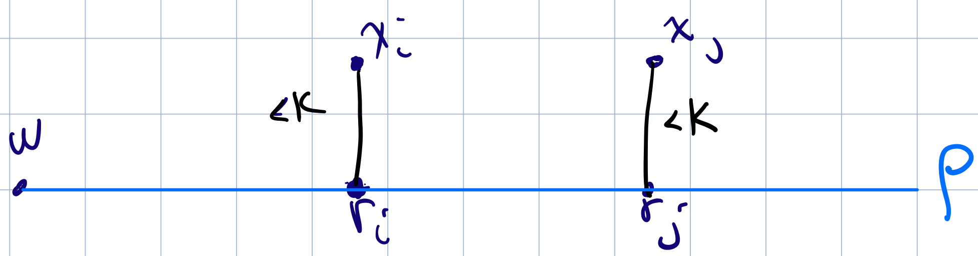
Let r = nearest point to x_i on σ_j, so $d(x_i, r) \leq K$
Then $d(1, r) \leq d(1, x_i) + K$, so $r \in p(N) \subset p$
ie $\{x_i\} \subseteq N_K(p)$.

② If $y \in p$, then $y \in \sigma_j$ for some j, so $\exists\, x_i$, $i \leq j$
$d(x_i, y) < K$, ie $p \subset N_K(\{x_i\})$

<u>Corollary</u> $\{x_i\} \longrightarrow \infty$ and $\{x_i\} \sim \{p(i)\}$

<u>Proof</u>

We have the following picture:



$$(x_i, x_j)_w \geq \min\{(x_i, r_i)_w, (x_j, r_i)_w\}$$
$$\geq \min\{(x_i, r_i)_w, (x_j, r_j)_w, (r_i, r_j)_w\}$$

Now

$$(x_i, r_i)_w \geq d(w, [x_i, r_i]) - 2\delta \geq d(w, x_i) - K - 2\delta \to \infty$$

Since $\{x_i\}$ is a quasi-geodesic, $d(w, x_i) \to \infty$,

So $(x_i, r_i)_w \geq d(w, [x_i, r_i]) - 2\delta \geq d(w, x_i) - K - 2\delta \to \infty$

and $(r_i, r_j)_w = \min(d(w, r_i), d(w, r_j))$

$$\geq \min(d(w, x_i) - K, d(w, x_j) - K) \to \infty$$

Therefore $(x_i, x_j)_w \longrightarrow \infty$ and $\{x_i\} \sim \{r_i\}$.

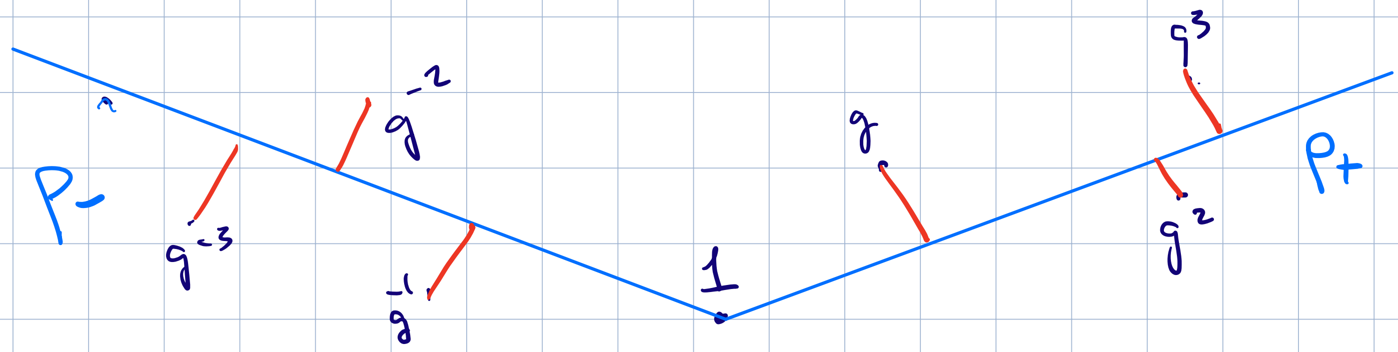
(Furthermore, $\{r_i\} \sim \{p(i)\}$ since $d(w, r_i)$ and $d(w, p(i))$ both $\to \infty$.) $\blacksquare$

So we may define $g_\infty = \{g^i\}_{i \geq 0}$

Next task: Show $g_\infty \neq (g^{-1})_\infty$,

ie $\{g^i\} \not\sim \{g^{-i}\}$

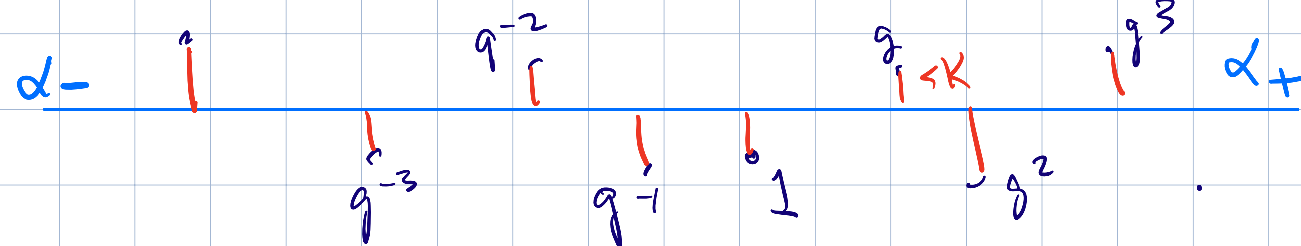
We know there are geodesic rays p^+ and p^- and a constant K.



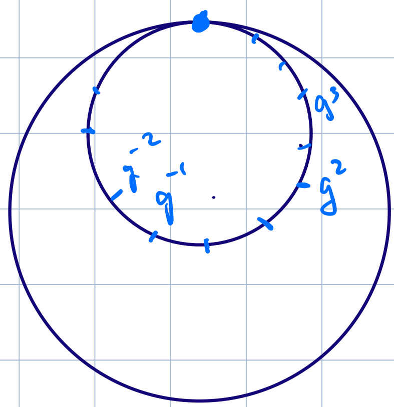
$d(g^i, p^+) \leq K$, $d(g^{-i}, p^-) \leq K$.

and $d(p^+(t), \{g^i\}) \leq K$ and $d(p^-(t), \{g^{-i}\}) \leq K$

I want to find an <u>axis</u> for g, ie a bi-infinite path α such that every segment is a geodesic, and a constant K such that $\alpha \subset N_K\{g^i\}$ and $\{g^i\} \subset N_K(\alpha)$



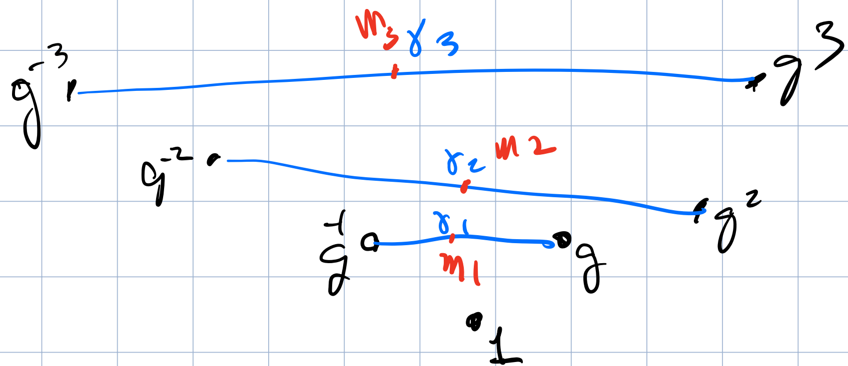
ie, I want to avoid the following picture:



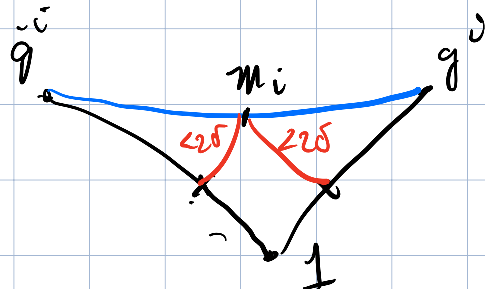
$\{g^i\}$ and $\{g^{-i}\}$ go to the same point at ∞!

Proposition There is a bi-infinite geodesic ray α such that $\{\alpha(i)\} \to g_\infty$ and $\{\alpha(-i)\} \to g_{-\infty}$.

Proof Let α_i be a geodesic from g^{-i} to g^i with midpoint m_i:

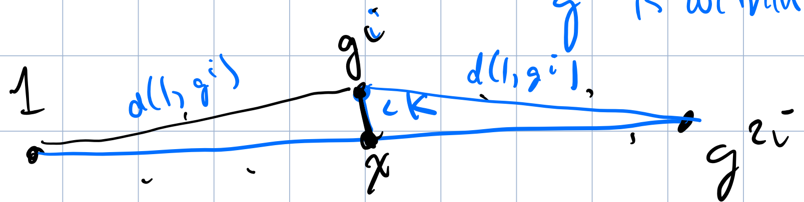


Look at the geodesic triangle with vertices g^{-i}, g^i and 1



Since this triangle is isosceles, m_i is the interior point on the side $[\bar{g}^i, g^i]$, so:

$$d(1, m_i) \leq (\bar{g}^i, g^i)_1 + 2\delta$$
$$= \tfrac{1}{2}\left(d(1,\bar{g}^i) + d(1,g^i) - d(\bar{g}^i, g^i)\right) + 2\delta$$
$$= \tfrac{1}{2}\left(d(1,g^i) + d(1,g^i) - d(1,g^{2i})\right) + 2\delta$$



g^i is within K of the geodesic from 1 to g^{2i}

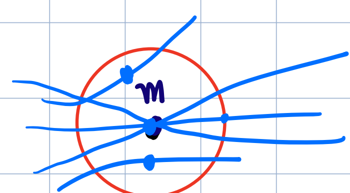
$$d(1,x) \geq d(1,g^i) - K$$
$$d(x, g^{2i}) \geq d(1,g^i) - K$$

so

$$d(1,g^{2i}) \geq d(1,g^i) + d(1,g^i) - 2K$$
$$d(1,g^i) + d(1,g^i) - d(1,g^{2i}) \leq 2K + 2\delta$$

so $d(1,m_i) \leq \tfrac{1}{2}(2K + 4\delta) = K + 2\delta.$

So the midpoint of <u>every</u> γ_i is in $B_{K+2\delta}(1)$. So an ∞ number of m_i are the same, call it m.



Passing to further subsequences, we may assume the γ_i agree on $B_1(m)$ then on $B_2(m)$, etc. Let $\alpha_n = \gamma_i \cap B_n(m)$, so $m \in \alpha_1 \subset \alpha_2 \subset \cdots$.

Set $\alpha = \bigcup \alpha_i$. Any segment of α is in some γ_N, so this is a bi-infinite geodesic, and there is a constant K such that $\{g^i\} \subset N_K(\alpha)$ and $\alpha \subseteq N_K\{g^i\}$.

<u>Corollary</u>: $g^{\infty} \neq g^{-\infty}$

<u>proof</u> There is arbitrarily large i such that the geodesic from $\bar{g}^i$ to g^i passes through m,

so $(\bar{g}^i, g^i)_m = 0$, so

$\{g^i\} \neq \{\bar{g}^i\}$.

To play ping-pong, we need another infinite-order element h, whose axis has endpoints $h_\infty, h_{-\infty}$ different from $g_{\pm\infty}$.

The existence of such an h follows from the fact that G has infinitely many ends, so the centralizer of $\langle g \rangle$ is not all of G. Let $h \in G \setminus C\langle g \rangle$.

We may assume neither g nor h is a proper power. (If $g = g_0^k$ or $h = h_0^k$ with $k \neq 0$, replace g by g_0, h by h_0.)

Assume in addition that G is torsion-free. (this is not necessary but simplifies the argument) Then h has infinite order, and

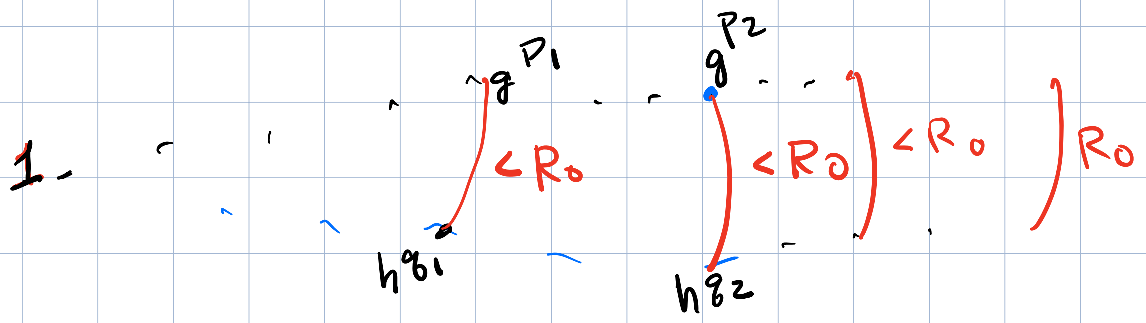
$$gh = hg \Leftrightarrow g^k h^l = h^l g^k \text{ for some } k, l \neq 0$$

We need to show $g_\infty \neq h_\infty$

<u>Lemma</u>: For any R_0, there are at most finitely many pairs (p,q) with $d(g^p, h^q) < R_0$

<u>Proof</u> Suppose there are infinitely many such pairs. Since $B_{R_0}(1)$ is finite,

among them we can find infinitely many pairs $(p_1,q_1), (p_2,q_2) \dots$ with $p_i \neq p_j$, $q_i \neq q_j$.



i.e. $d(1, g^{p_i} h^{-q_i}) < R_0$ for all i

Finiteness of $B_{R_0}(1)$ now implies that for some $i \neq j$

$$g^{p_i} h^{-q_i} = g^{-p_j} h^{-q_j}$$

$$\Rightarrow h^{q_j - q_i} = g^{p_i - p_j},$$

so $h^{q_j - q_i}$ and $g^{p_i - p_j}$ are non-trivial and commute,

so $gh = hg$. $\times$

So to prove $g_\infty \neq h_\infty$, it suffices to prove the following Lemma:

<u>Lemma</u>: If $g_\infty = h_\infty$, then for some $R_0 > 0$

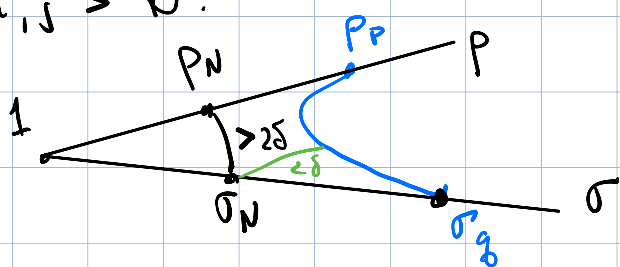
there are ∞ many pairs p, q with $d(g^p, h^q) < R_0$

Proof Let ρ and σ be geodesic rays starting at 1, within distance K of $\{g^i\}$ and $\{h^i\}$. Then

$$\rho_\infty = g_\infty = h_\infty = \sigma_\infty .$$

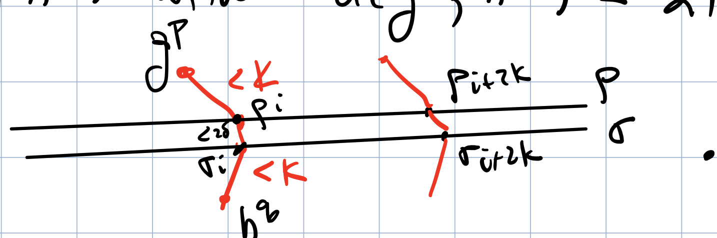
Let ρ_i and σ_i denote the vertices on ρ and σ at distance i from 1. We claim that $d(\rho_i, \sigma_i) < \delta$ for all i.

If not, fix N with $d(\rho_N, \sigma_N) = D > 2\delta$. Then $(\rho_i, \sigma_j)_1 \leq d(1, [\rho_i, \sigma_j]) \leq N + \delta$ for all $i, j > N$:



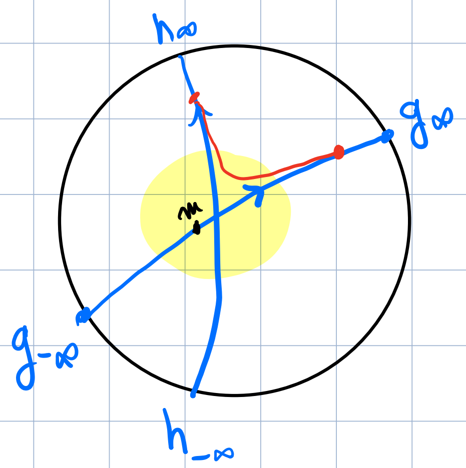
This contradicts our assumption that $\rho_\infty = \sigma_\infty$, ie $(\rho_i, \sigma_i)_1 \to \infty$.

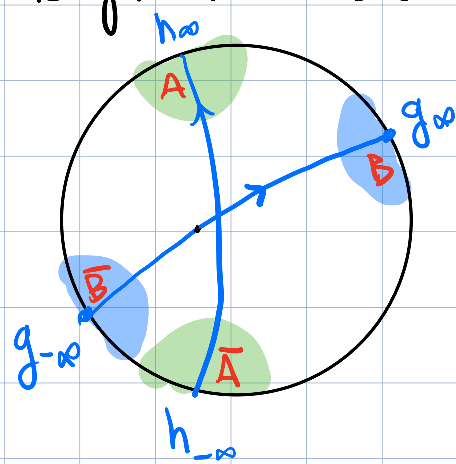
For each i, there is at least one g^p within distance K of ρ_i, and at least one h^q within distance K of σ_i. Therefore we can find infinitely many distinct pairs (g^p, h^q) with $d(g^p, h^q) \leq 2K + 2\delta$.



We now have:

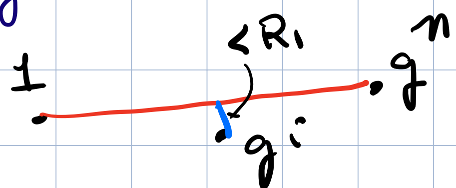
Now we need the neighborhoods!



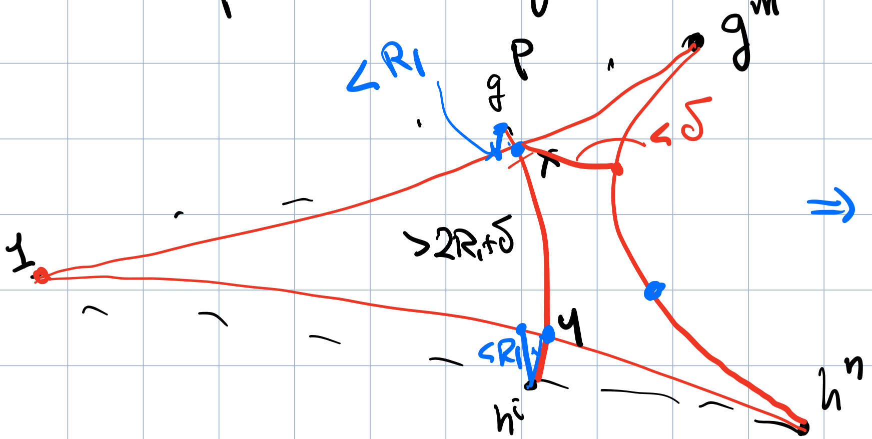


Lemma: $\exists R>0$ s.t. $\forall m, n$ the geodesic $[g^m, h^n]$ intersects $B_R(1)$

Proof: $\{g^i\}$ a quasigeodesic $\Rightarrow \exists R_1$ s.t. $\forall i, n$, $i<n \Rightarrow d(g^i, [1, g^n]) < R_1$:



Choose p s.t. $d(g^p, \{h^i\}) > 2R_1 + \delta$



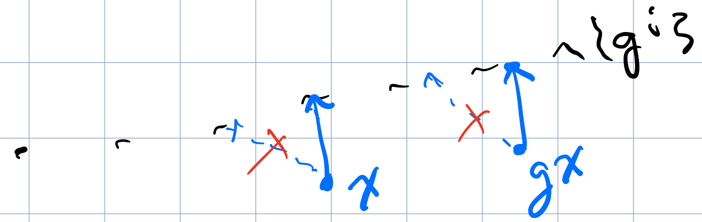
$\Rightarrow d(x,y) > \delta$

Then $d(1, [g^m, h^n]) < \underbrace{d(1, g^p) + \delta}_{R.}$ ✓

Lemma For any $x \in X$

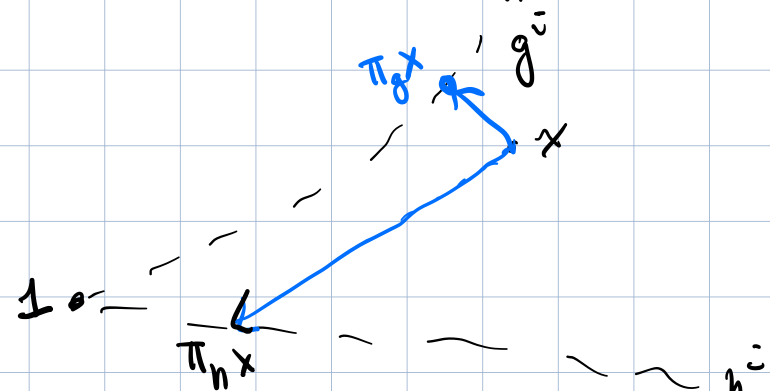
Let $\pi_g x$ = nearest point on $\{g^i\}$ to x.

Make the choice invariant under G-action

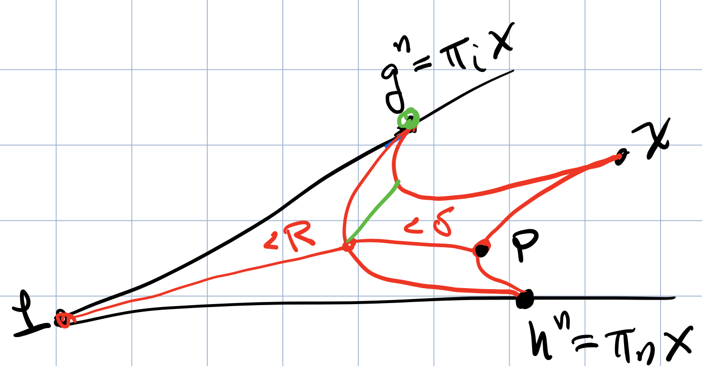


Define $\pi_h x$ similarly.

Claim: For some $M > 0$, either $d(1, \pi_g x) < M$ or $d(1, \pi_h x) < M$



Proof

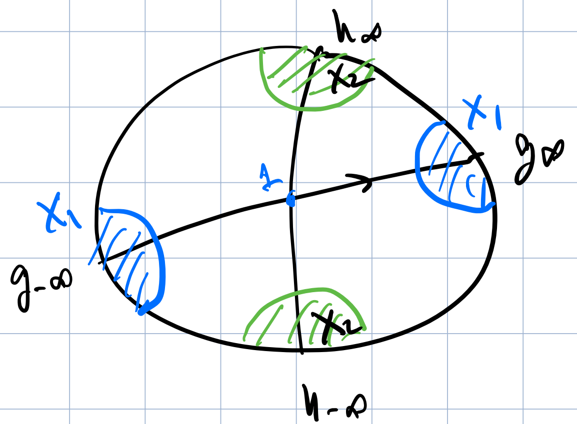


$$d(1, \pi_h x) \le d(1,p) + d(p, \pi_h(x))$$
$$= d(1,p) + d(p, \pi_h(p))$$
$$\le d(1,p) + d(p,1) = \underline{2d(1,p)}$$

M

Now we can prove the theorem.

Let $X_1 = \{x \mid \pi_g x = g^m,\ d(1, g^m) > M\}$
$(\Rightarrow d(1, \pi_h(x)) < M)$
$X_2 = \{x \mid \pi_h x = h^n,\ d(1, h^n) > M\}$



$X_1 \cap X_2 = \emptyset.$

$\forall x_2 \in X_2,\ d(1, \pi_h x_2) > M$
$\Rightarrow d(1, \pi_g x_2) < M$

$$\pi_g(g^m x_2) = g^m \pi_g(x_2)$$

$\Rightarrow$

$$d(1, \pi_g g^m x_2) \geq d(1, g^m) - d(1, \pi_g x_2)$$
$$\geq d(1, g^m) - M$$

Similarly $\forall x \in X_1$
$$d(1, \pi_h(h^n x_2)) \geq d(1, h^n) - M.$$

So if m, n are big enough $d(1, g^m), d(1, h^n) > 2M$

so $g^m X_2 \subset X_1$
and $h^n X_1 \subset X_2$

So can play ping-pong !!

and see $\langle g^m, h^n \rangle \cong F_2$ ✓

What is known about Hyperbolic groups G?

We showed:

- If G is infinite it has an element of infinite order
- The centralizer of an infinite order element g is a finite extension of $\langle g \rangle$
  (which implies G doesn't contain $\mathbb{Z}^2$)
- If G has more than 2 ends it contains a copy of F_2
- G has only finitely many conjugacy classes of finite elements (in the Exercises)
- G is finitely presented, in fact has a finite presentation with a linear Dehn function.

Other things people have proved:

- The Dehn function of <u>any</u> finite presentation for G is linear
- If G has a subquadratic Dehn function then G is hyperbolic
- If G has more than 2 ends, it contains an infinite normal subgroup H with infinite quotient G/H
  (G is <u>really</u> not simple!)

- G has solvable word, conjugacy and isomorphism problems
  (The isomorphism problem is particularly difficult to solve - was first proved by Sela for torsion-free hyperbolic groups)

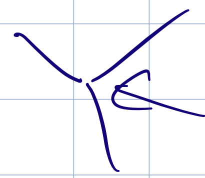
- The homology $H_i(G)$ is finite-dimensional for all i (this generalizes the fact that G is finitely presented, since finitely presented G have finite-dimensional $H_2(G)$.)

- If G has more than 2 ends, then the number of elements of length n (in any finite generating set) grows exponentially with n

- A random presentation gives a hyperbolic group, for a suitable notion of "random."

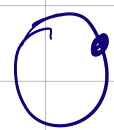
More information about hyperbolic groups can be found in Gromov's original article or in Bridson-Haefliger's book.

$BV - H^*(H) =$



BV generators have a weight

$BV = H_*$ of framed little disk operad



"Formality of BV-operad." has a problem.

BV $\hookrightarrow$ Koszul dual operad.

DBV = space of decorated BV-trees.

Trees $\to$ decora

BV_0 is trivial.

vertices $\to$ trees, decorated by BV

weight 0 = framed graph

weight = BV algebra — degrees of generators.

$= H_*(LD)$

degree.

Koszul-duality = free generation.

Koszul duality –

$\chi(DBV)$
$\chi(BV)$ — same euler char in dual weights

conj: $H^*(DBV) \cong H^*(BV)$

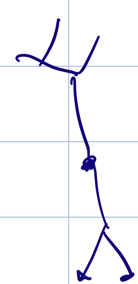
Make an actual complex that computes
H^* (handlebody gp)
explain weight filtration geometrically.

How can you have such a complex?
concentrated, but differential retains some weights.

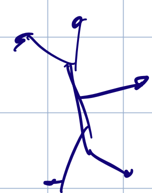
---

weight filtration on OS:
difference between graded operads
and the graph operad

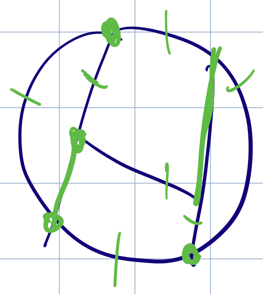
$(\ell_1-2)+(\ell_2-2)+2$
$= \ell_1+\ell_2$
$\ell_1+\ell_2$
-4
1



Decorated graphs $\leftrightarrow$ # of leaves



deg -2



Total # of leaves =

non-tree.

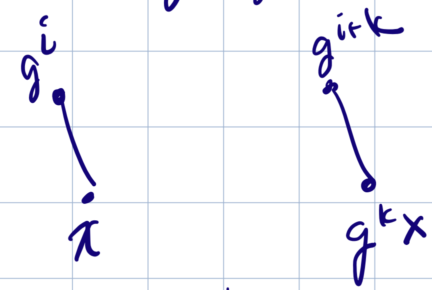
For any $x \in X$, let

$\pi_g x$ = a nearest point on $\{g^i\}$ to x

there are only finitely many choices.

Make them equivariantly, i.e. set

$\pi_g(g^k x) = g^k \pi_g x$

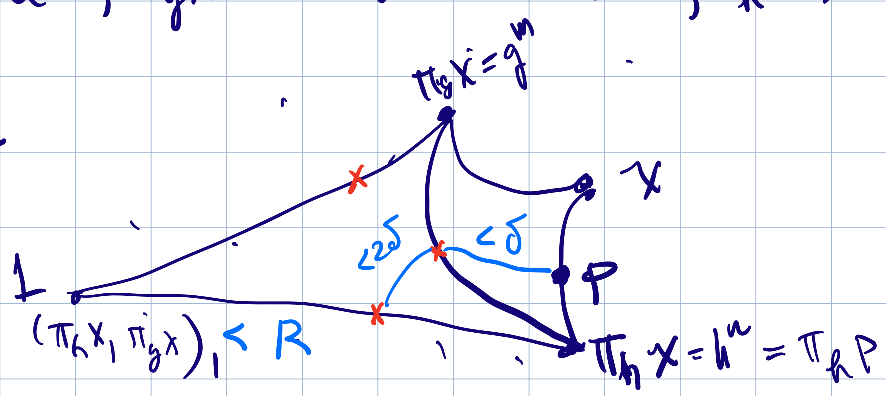


Define $\pi_h x$ similarly

<u>Lemma</u> There is $M < 0$ such that either

$d(1, \pi_g x) < M$ or $d(1, \pi_h x) < M$

<u>Proof</u>



$$
\begin{aligned}
d(1, \pi_h x) &\le d(1, p) + d(p, \pi_h x) \\
&\le R + 3\delta + d(p, \pi_h p) \\
&\le R + 3\delta + 3\delta = R + 6\delta \\
&= M
\end{aligned}
$$

<u>Now</u>: Let $A = \{x \mid \pi_g x = g^n,\ d(1, g^n) > M\}$

$B = \{x \mid \pi_h x = h^n,\ d(1, h^n) > M\}$

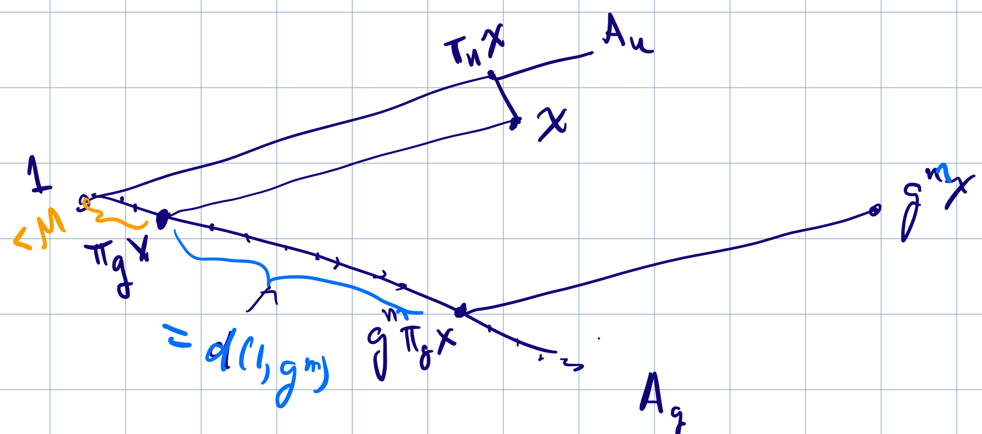
Then $x \in B \Rightarrow x \notin A$ by the lemma

ie $d(1, \pi_g x) \leq M$

let $x \in B$. $\pi_g(g^m x) = g^m \pi_g x$

$\Rightarrow d(1, \pi_g g^m x) = d(1, g^m \pi_g x) \geq d(1, g^m) - d(1, \pi_g x)$

$\geq d(1, g^m) - M$



so for m large enough, $d(1, \pi_g(g^m x)) > M$

$\Rightarrow g^m x \in A$.

Similarly, for n large enough, $x \in B \Rightarrow h^n x \in A$.

$\Rightarrow \langle g^m, h^n \rangle$ generate a free group.

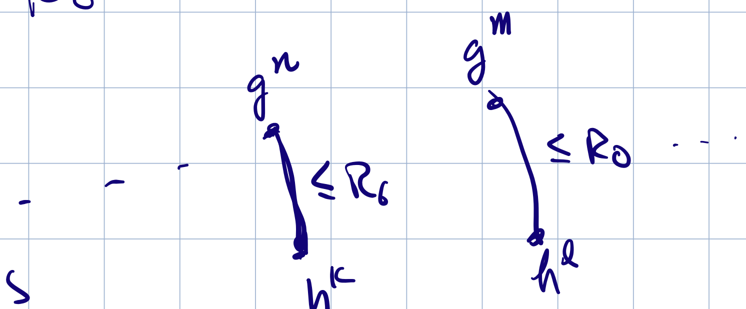
Still need to justify $h_\infty \neq g_\infty, g_{-\infty}$.

We know $\exists a$, $ga \neq ag$. $h = aga^{-1}$ has ∞ order. since g does.

Simplifying assumptions: $hg \neq gh$ and, in fact $g^k h^\ell \neq h^\ell g^k$ for any $k, \ell \neq 0$.

<u>Lemma</u>: $\forall R_0$, $\exists N$ s.t. $n > N \Rightarrow$ $d(g^n, \{h^i\}) > R_0$

<u>pf</u> If not



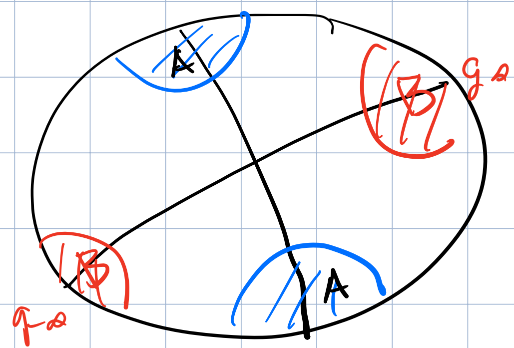
$\exists$ ∞ many quadruples (n,k,m,ℓ) with $n \neq m$, $k \neq \ell$ s.t. $d(g^n, h^k) < R_0$ and $d(g^m, h^\ell) < R_0$

$$\Rightarrow d(1, g^{-n}h^k) < R_0$$
and $$d(1, g^{-m}h^\ell) < R_0$$

$\Rightarrow$ for some $n \neq m$, $k \neq \ell$
$$g^{-n} h^k = g^{m} h^\ell$$

$\Rightarrow h^{k-\ell} = g^{m-n} \Rightarrow h^{k-\ell}$ and g^{m-n} commute $\times$.

So $\hat{X}$ makes a good ping-pong table:
Given g, h with $g^{\pm\infty} \neq h^{\pm\infty}$



For suff large N,
$g^N A \subset B$
$h^N B \subset A$
$\Rightarrow \langle g^N, h^N \rangle \cong F_2$

If $G = \mathbb{Z}$ (or contains $\mathbb{Z}$ as a subgp of finite index
ie is virtually $\mathbb{Z}$)
then G is hyperbolic but doesn't contain F_2
and if G is finite, it is hyperbolic " " "
These are called elementary hyperbolic groups

Thm A hyperbolic group is either cyclic or contains an F_2.

Jacques Tits proved a linear group (eg matrix group) is either virtually solvable or contains an F_2.
This is now called a "Tits alternative"
So hyperbolic groups satisfy a (very strong) Tits alternative