$\boxed{\text{CAT(0) metric spaces.}}$

Many of the most interesting groups are <u>not</u> hyperbolic

Sometimes this can be easily seen because they contain a copy of $\mathbb{Z}^2$ – eg

- $SL(n, \mathbb{Z})$ for $n \geq 3$, and other lattices in semi-simple Lie groups
- Mapping class groups of compact surfaces of genus ≥ 2.
- Automorphism groups of free groups

Sometimes there are more subtle reasons, eg
- Baumslag-Solitar groups $BS(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle$, $n \geq 2$

We would still like to study them using the Svarc-Milnor lemma and curvature constraints.

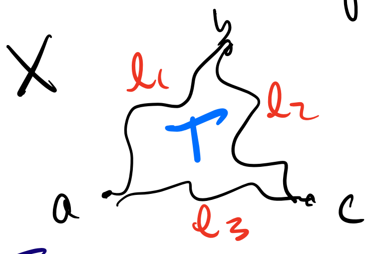
A notion of curvature for a general metric space was studied by Gromov, based on work by Caratheodory, Alexandrov and Toponogrov,

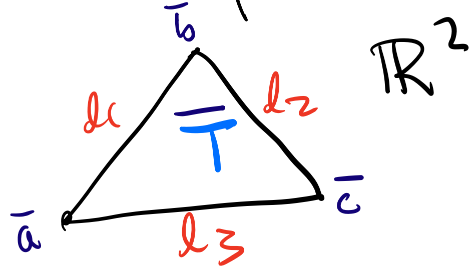
so he called it CAT(K), where $K \in \mathbb{R}$ is the "curvature".

In particular, CAT(0) is a notion of <u>non-positive</u> curvature

It is also defined in terms of triangles, but is "less fuzzy" than the notion of δ-thin triangles used in the definition of hyperbolicity.

Let X be a geodesic metric space



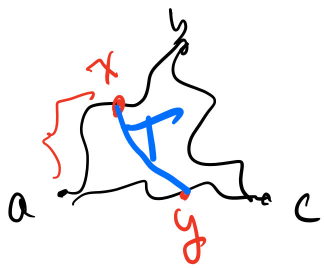


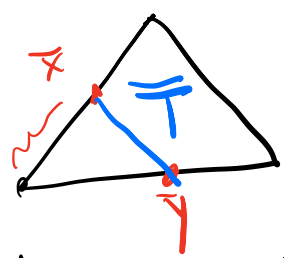
T = geodesic triangle

l_1, l_2, l_3 satisfy triangle inequality

$\bar{T}$ = triangle in $\mathbb{R}^2$ with same side lengths. "comparison triangle"

Def X = proper geodesic metric space is CAT(0) if every geodesic triangle T is thinner than $\bar{T}$, ie for $x, y \in T$, let $\bar{x}$, $\bar{y}$ be the corresponding points in $\bar{T}$:

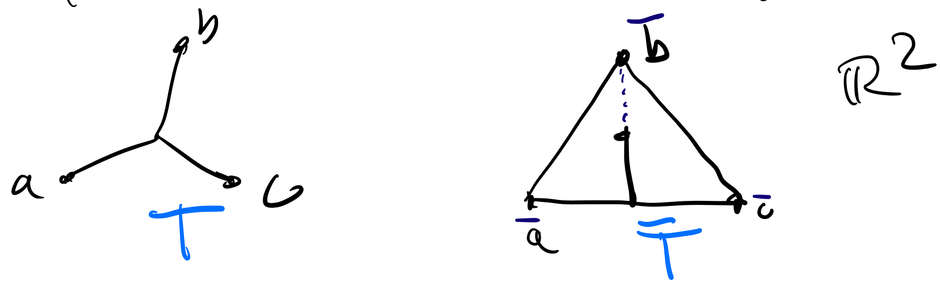




then $d_X(x,y) \le d_{\mathbb{R}^2}(\bar{x}, \bar{y})$

Example: $\mathbb{H}^2$ is CAT(0)

Example: A tree is CAT(0)



Example: $\mathbb{R}^n$ is CAT(0)

Exercise: If X, Y are CAT(0) then $X \times Y$ is, where

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

In particular, products $T_1 \times T_2$ of trees are CAT(0)

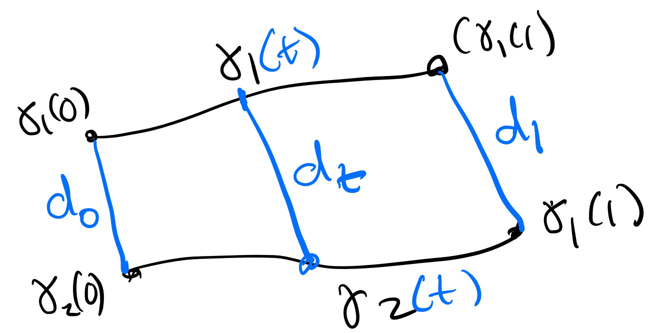
Theorem X CAT(0) $\Rightarrow$ there is a unique geodesic between any two points.

Lemma: X CAT(0), $\gamma_1, \gamma_2 : [0,1] \to X$ geodesics then

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)\, d(\gamma_1(0), \gamma_2(0)) + t\, d(\gamma_1(1), \gamma_2(1))$$

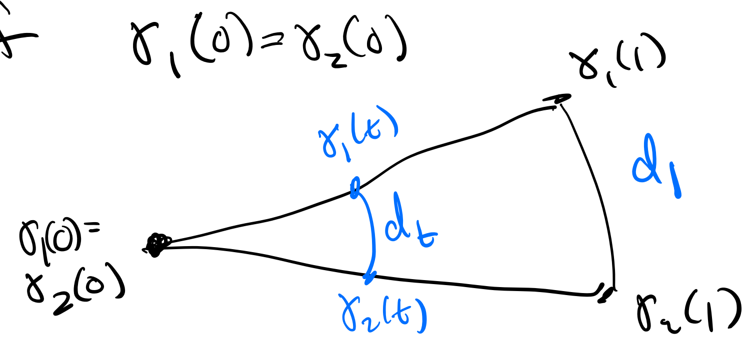
Proof:

Picture:

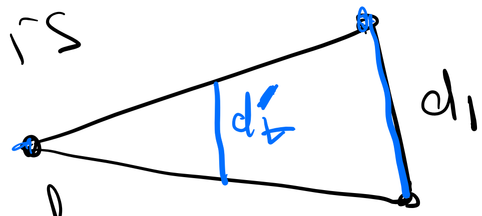


We want: $d_t \leq (1-t)d_0 + t \cdot d_1$

If $\gamma_1(0) = \gamma_2(0)$

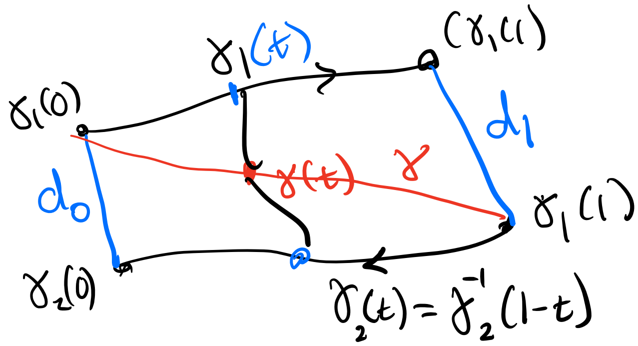


The comparison triangle is



So $d_t \le d_t' = t \cdot d_1$

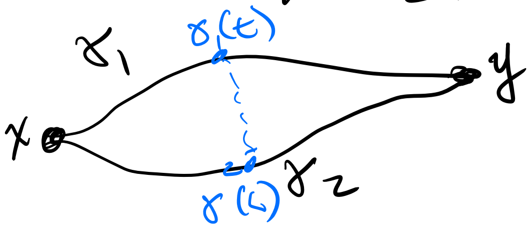
In general



$$d_t = d(\gamma_1(t), \gamma_2(t)) \le d(\gamma_1(t), \gamma(t)) + d(\gamma(t), \gamma_2(t))$$
$$\le t\, d_1 + (1-t)\, d_0 \quad \checkmark$$

The theorem is now a corollary:

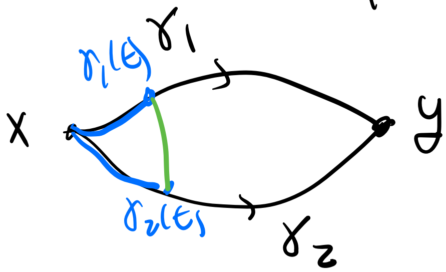
Let γ_1, γ_2 be two geodesics from x to y. Then $\gamma_1 = \gamma_2$.

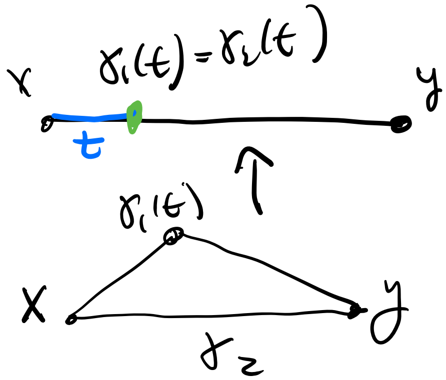


Pf

$$d(\gamma_1(t), \gamma_2(t)) \le t\, d_1 + (1-t)\, d_0$$
$$= t \cdot 0 + (1-t) \cdot 0 = 0. \checkmark$$

Note: Easier proof:





Look at comparison Δ for

Get $d(\gamma_1(t), \gamma_2(t)) \leq 0$

$\Rightarrow \gamma_1(t) = \gamma_2(t)$.

We can use this to prove the following:

Theorem: A CAT(0) metric space X is contractible

We need

Lemma: Geodesics vary continuously with their endpoints.

ie $x_n \longrightarrow x$ $\quad y_n \longrightarrow y$

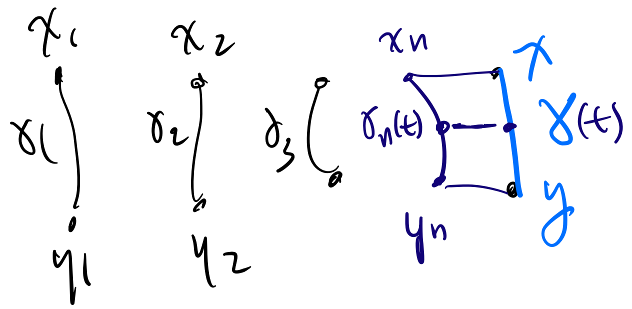
let $\gamma_n : [0,1] \to X$ be the geodesic x_n to y_n

$\gamma : [0,1] \to X$ the geodesic x to y

Then $\{\gamma_n\} \longrightarrow \gamma$

<u>Proof</u> Convergence here means in the compact-open topology.

The previous lemma guarantees pointwise convergence



$x_n \to x \qquad y_n \to y \qquad \Rightarrow \gamma_n(t) \to \gamma(t)$

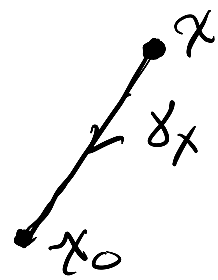
Then compactness of $[0,1]$ allows us to apply the Arzela-Ascoli theorem to get uniform convergence.

Proof of theorem (contractibility):

Fix $x_0 \in X$

For any $x \in X$,

let γ_x be the unique geodesic x to x_0



Define $F : X \times [0,1] \longrightarrow X$

$$(x, t) \longmapsto \gamma_x(t)$$

This is a continuous map.

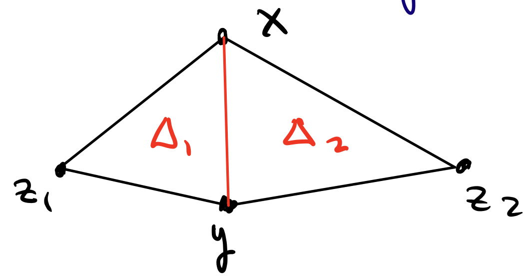
$F(0) = id$

$F(1) \equiv x_0$

ie F is a homotopy from X to $\{x_0\}$ ✓

We can get new CAT(0) spaces from old ones by a gluing procedure, based on the following Lemma

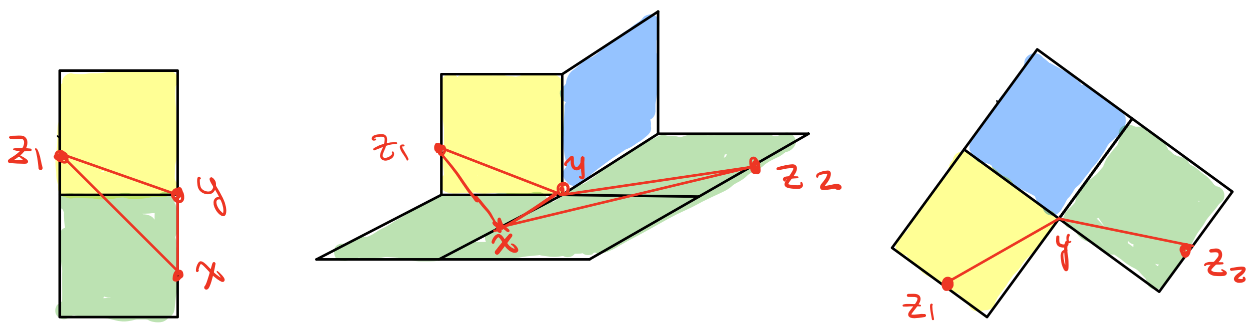
Alexandrov Lemma Let $\Delta(x, z_1, z_2)$ be a geodesic triangle with vertices x, z_1, z_2 and y a point on the geodesic $[z_1, z_2]$

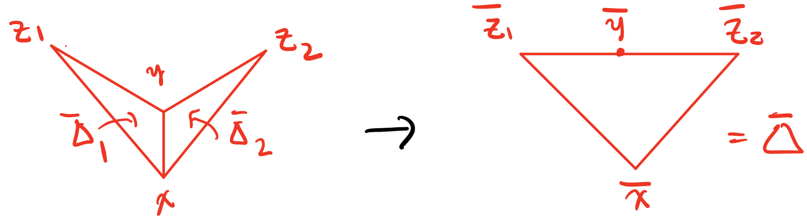


$\Delta_1 = \Delta(x, y, z_1)$, $\Delta_2 = \Delta(x, y, z_2)$

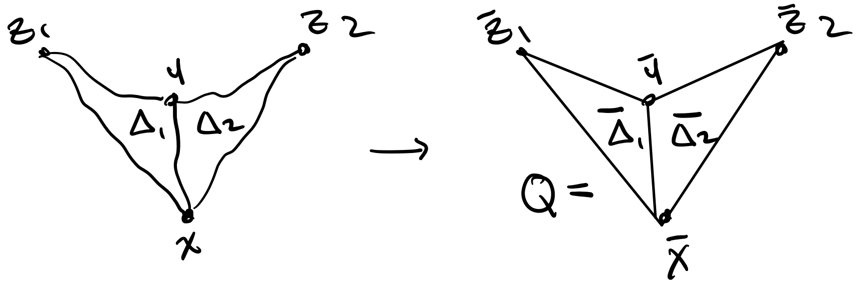
If Δ_1 and Δ_2 satisfy the CAT(0) condition, then so does $\Delta(x, z_1, z_2)$.

Example to keep in mind: hallway corner



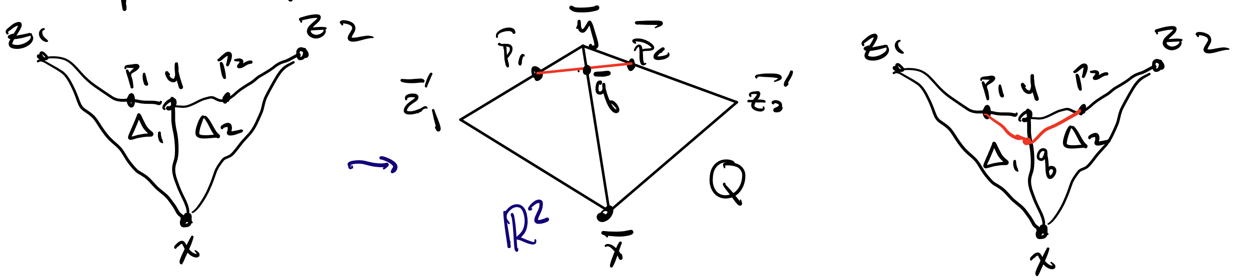


Proof: Look at the quadrilateral Q formed by $\bar{\Delta}_1$ and $\bar{\Delta}_2$ in $\mathbb{R}^2$:



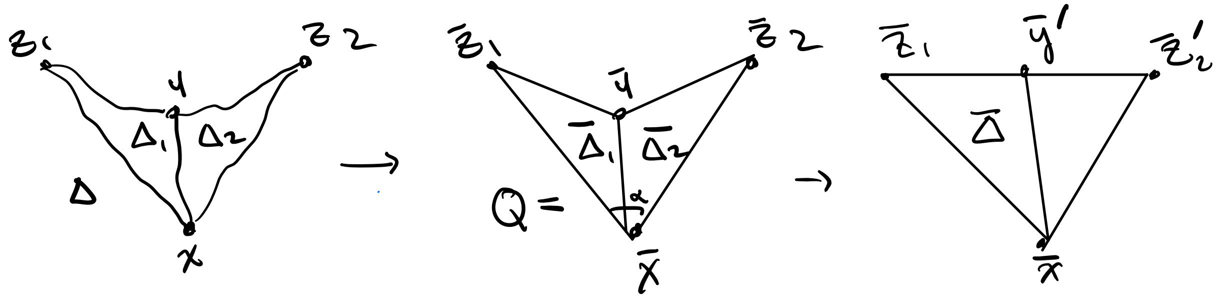
Claim $\angle(\bar{z}_1 \bar{y} \bar{x}) + \angle(\bar{z}_2 \bar{y} \bar{x}') \geq \pi$

Proof: Suppose it is $< \pi$. Choose $p_i \in [z_i, y]$ close to y,



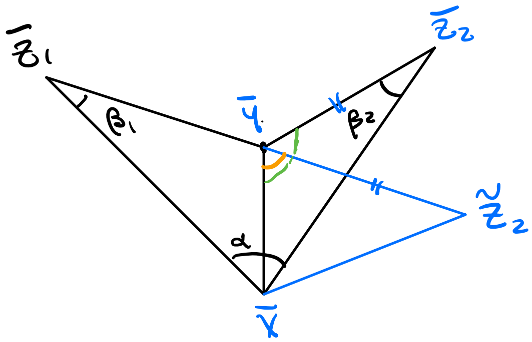
Then
$$\begin{aligned} d_X(p_1,p_2) &= d_X(p_1,y) + d_X(y,p_2) \\ &= d_{\mathbb{R}^2}(\bar{p}_1,\bar{y}) + d_{\mathbb{R}^2}(\bar{y},\bar{p}_2) \\ &> d_{\mathbb{R}^2}(\bar{p}_1,\bar{q}) + d_{\mathbb{R}^2}(\bar{q},\bar{p}_2) \\ &\geq d_X(p_1,q) + d_X(q,p_2) \\ &\geq d_X(p_1,p_2) \quad \divideontimes \end{aligned}$$

Now we straighten Q to obtain the comparison triangle $\bar{\Delta}$ for Δ:



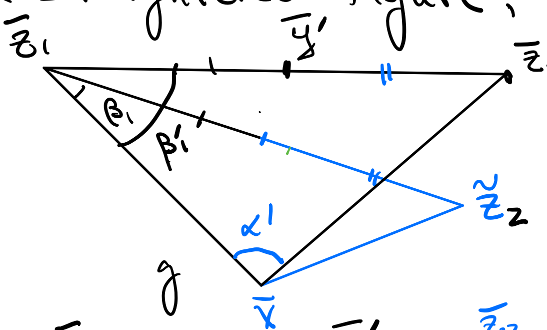
<u>Claim</u>: straightening increases the angles at $\bar z_1$, $\bar z_2$ and $\bar x$

<u>Proof</u>: continue the edge $\bar z_1 \bar y$ and mark $\tilde z_2$ on it with $d(\bar y, \tilde z_2) = d(y, \bar z_2)$:



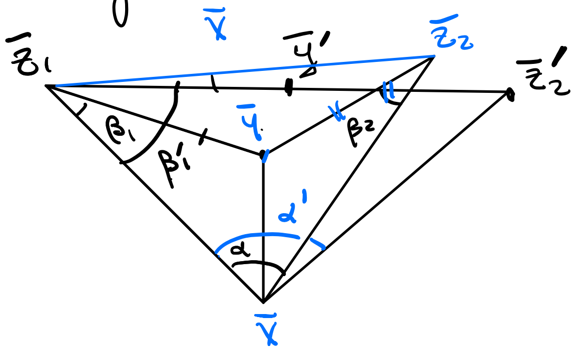
$\angle \bar x \bar y \tilde z_2 \leq \angle \bar x \bar y \bar z_2$
$\Rightarrow d(\bar x, \tilde z_2) \leq d(\bar x, \bar z_2)$
(law of cosines)

in the straightened figure, this implies $\beta_1' \geq \beta_1$



(law of cosines again)

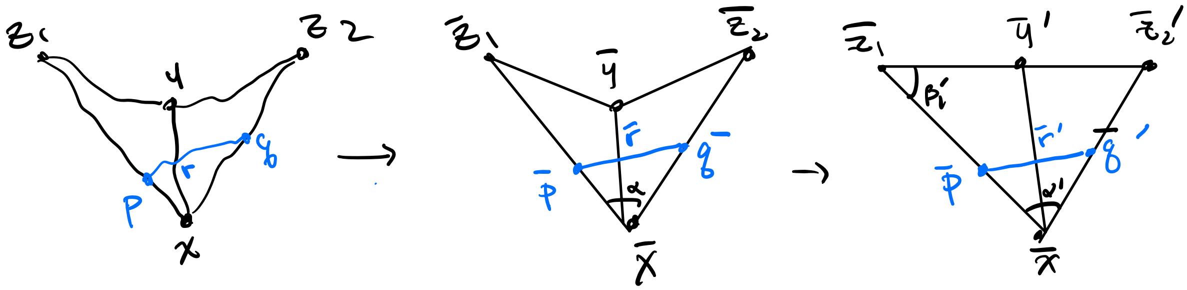
and



$d(\bar z_1, \bar z_2) \leq d(\bar z_1, \bar y) + d(y, \bar z_2) = d(\bar z_1, \bar z_2') \Rightarrow \alpha < \alpha'$

Now for the CAT(0) inequality:

If $[\bar{p},\bar{q}]$ stays in Q, this is clear: eg:



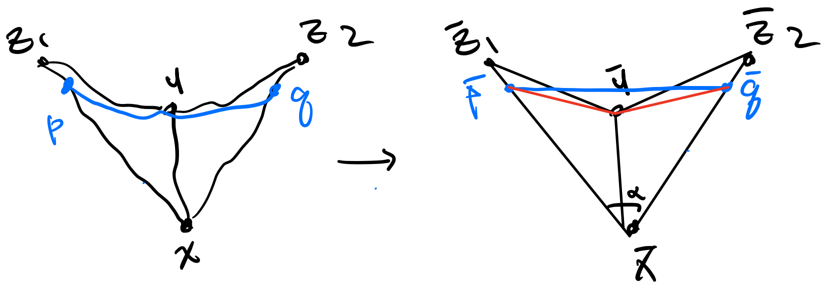
$$d(p,q) = d(p,r) + d(r,q) \leq d(\bar{p},\bar{r}) + d(\bar{r},\bar{q})$$

(because Δ_1, Δ_2 are CAT(0))

$$= d(\bar{p},\bar{q})$$

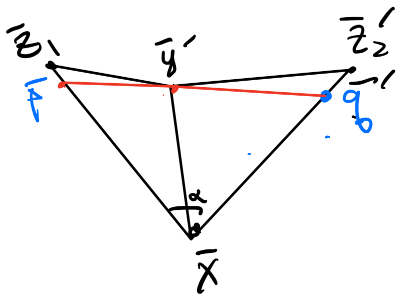
$$\leq d(\bar{p},\bar{q}') \text{ since } \alpha' \geq \alpha$$

If $[\bar{p},\bar{q}]$ leaves Q:



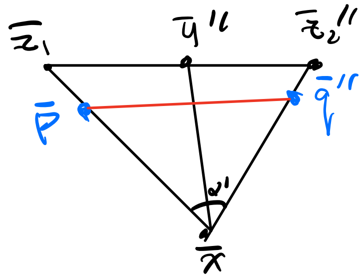
then straighten

until it passes through $\bar{y}'$



The red distances are larger since β_1 and β_2 are larger.

Continue to straighten; then $d(\bar{p}, \bar{q}')$ grows since α grows, as before.



So, in the end
$$
\begin{aligned}
d(p,q) &= d(p,y) + d(y,q) \\
&\le d(\bar{p}, \bar{y}) + d(\bar{y}, \bar{q}) \\
&\quad (\text{because } \Delta_1, \Delta_2 \text{ are } CAT(0)) \\
&\le d(\bar{p}, \bar{q}') \\
&\le d(\bar{p}, \bar{q}'') \checkmark
\end{aligned}
$$

Easy corollary:

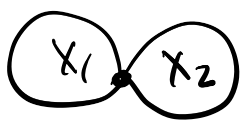
X_1, X_2 CAT(0), $Y_1 \subset X_1$, $Y_2 \subset X_2$

Y_i convex, both $\cong Y$.

Then $X_1 \cup_Y X_2$ is CAT(0)

Pf.: Exercise

eg X_1, X_2 CAT(0) $X_1 \vee X_2$



is CAT(0)

$\boxed{\text{CAT(0) groups}}$

<u>Definition</u> A group G is CAT(0) if it acts properly and cocompactly on a CAT(0) space X.

Recall that CAT(0) metric spaces are assumed to be proper, so we may apply Svarc-Milnor to conclude that G is quasi-isometric to X

Unlike hyperbolicity, the CAT(0) property is not a quasi-isometry invariant, so there is no obvious space to use to determine whether G is CAT(0)

(It is an open problem whether every hyperbolic group is also CAT(0).)

If G_1 and G_2 are CAT(0),

$G_1 \curvearrowright X_1$, $G_2 \curvearrowright X_2$ cocompact, proper

$\Rightarrow$ $G_1 \times G_2 \curvearrowright X_1 \times X_2$ is cocompact and proper

By your exercise,

X_1, X_2 CAT(0) $\Rightarrow$ $X_1 \times X_2$ CAT(0), so

$G_1 \times G_2$ is also CAT(0)

We can use Alexandrov's lemma to prove that free products $G_1 * G_2$ are CAT(0) as follows.

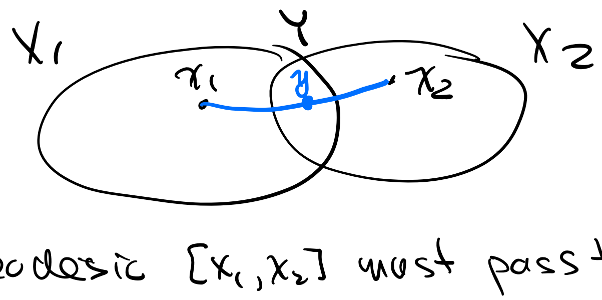
First note Alexandrov can be used to prove:

Prop Let X_1, X_2 be CAT(0) spaces with isometric convex subspaces $Y_1 \subset X_1$, $Y_2 \subset X_2$ $(Y_1 \cong Y_2 \cong Y)$

Then $X_1 \cup_Y X_2$ is CAT(0)

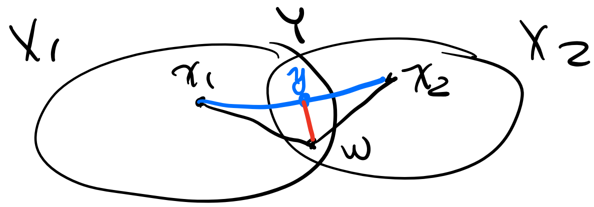
Proof Let $\Delta(x_1, x_2, w)$ be a geodesic triangle in $X_1 \cup_Y X_2$

If $\Delta \subset X_1$ or $\Delta \subset X_2$ it is CAT(0), so assume $x_1 \in X_1 \setminus Y$, $x_2 \in X_2 \setminus Y$

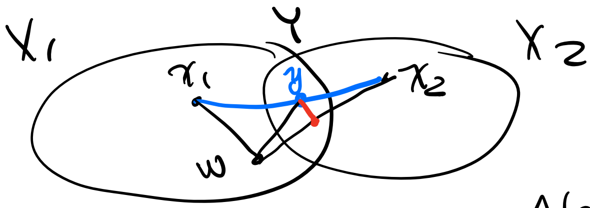


The geodesic $[x_1, x_2]$ must pass through Y, so contains some $y \in Y$

If $w \in Y$, then by convexity $[w, y] \subset Y$, so Alexandrov $\Rightarrow$ $\Delta(w, x_1 x_2)$ is CAT(0)



If $w \in X_1$, then $\Delta(w y x_2)$ is CAT(0) by the same argument, so again Alexandrov $\Rightarrow$ $\Delta(x_1 w x_2)$ is CAT(0)



(similar if $w \in X_2$) ✓

A space is locally CAT(0) (also called NPC - for "non-positively curved") if every point has a CAT(0) neighborhood

<u>Theorem</u> If Y is NPC, its universal cover $\tilde{Y}$ is a complete CAT(0) space.

The proof occupies a chapter in B-H (Chapter II.4 - The Cartan-Hadamard theorem) - we will not prove it.

So if Y is NPC <u>and</u> compact, $\pi_1(Y)$ is CAT(0).

Now suppose G_1 and G_2 are torsion-free CAT(0). Then $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ are free and proper, so $X_1 \longrightarrow X_1/G$ and $X_2 \longrightarrow X_2/G$ are covering spaces.

Since X_1, X_2 are contractible they are the universal covers of $Y_i = X_i/G_i$ so $\pi_1(Y_i) = G_i$

Furthermore, $Y_1 \vee_y Y_2$ is locally CAT(0): the only point to check is y.
If U_1 is a CAT(0) nbd of y in Y_1 and U_2 is a CAT(0) nbd in Y_2, then $U_1 \cap U_2 = \{y\}$

and $U_1 \cup U_2$ is CAT(0) by the Proposition.

$\therefore G_1 * G_2 = \pi_1(Y_1 \vee_y Y_2)$ acts on $\widetilde{(Y_1 \vee_y Y_2)}$, which is CAT(0) by the Cartan-Hadamard theorem, so $G_1 * G_2$ is CAT(0)

Properties of CAT(0) groups.

CAT(0) groups share a number of "tameness" properties with hyperbolic groups, but to prove them usually requires different methods

For example

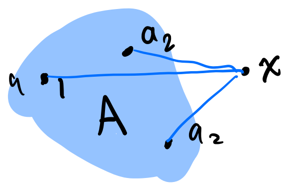
<u>Theorem</u> A CAT(0) group G has only finitely many conjugacy classes of finite subgroups.

This is also true for hyperbolic groups.
You proved this in the exercises for finite <u>cyclic</u> subgroups of hyperbolic groups, using the fact that long loops have uniformly short segments that are not geodesics.
But that's not true for CAT(0) spaces.

We will use a different idea:
First show any finite subgroup H of G fixes a point of the CAT(0) space X.
Then use properness of the action to reach the conclusion.

The fixed point will be the "center" of an orbit $H \cdot x_0$, which is a finite set. When you act by $h \in H$, the orbit Hx_0 doesn't move, so it's center doesn't move.

Let A be a finite set. To find a center:



For any $x \in X$, define

$$r(x,A) = \max_{a \in A} d(x,a) \qquad (\text{so } A \subset \overline{B}_r(x))$$

And $$r(A) = \inf_{x \in X} r(x,A)$$

If $r(x,A) = r(A)$, x is called a circumcenter for A

<u>Proposition</u> Every finite set A has a unique circumcenter.

To prove this, we will prove an inequality relating $r(x,A)$, $r(y,A)$, $d(x,y)$ and $r(A)$:

$$\boxed{d(x,y)^2 \leq 2\left(r(x,A)^2 + r(y,A)^2 - 2r(A)^2 \right)}$$

This immediately gives uniqueness:

If $r(x,A)=r(A)$ and $r(y,A)=r(A)$
then $d(x,y)=0$

To get existence, take a sequence $\{x_n\}$ of points in X such that
$$r(x_n,A) \longrightarrow r(A)$$

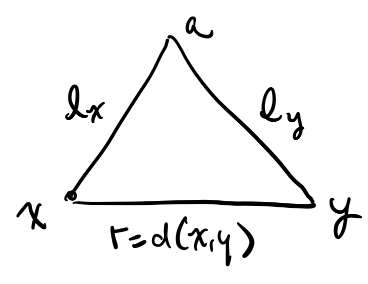
Then $d(x_i,x_j)^2 \leq 2\left(r(x_i,A)^2 + r(x_j,A)^2 - 2r(A)\right)$
$\longrightarrow 0$ as $i,j \to \infty$

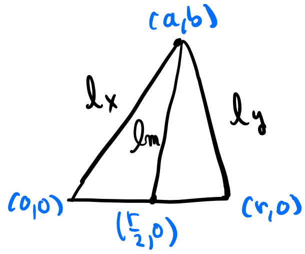
So $\{x_n\}$ is a Cauchy sequence

The Cartan-Hadamard theorem implies that a CAT(0) space with a proper cocompact group action must be complete,

so $\{x_n\} \longrightarrow x \in X$
and $r(x_n,A) \longrightarrow r(x,A)$ ✓

To prove the boxed inequality, form a geodesic triangle with vertices x, y and some $a \in A$
then consider the comparison triangle in $\mathbb{R}^2$





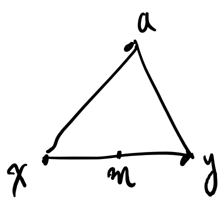
$$l_x = \sqrt{a^2+b^2} \qquad l_y = \sqrt{(a-r)^2+b^2}$$

$$l_m = \sqrt{(a-\tfrac{r}{2})^2+b^2}$$

So
$$\tfrac{1}{2}(l_x^2+l_y^2) = \frac{1}{2}\left(a^2+b^2+(a^2-2ar+r^2+b^2)\right)$$
$$= a^2+b^2-ar+\frac{r^2}{2}$$

and
$$l_m^2 = a^2-ar+\frac{r^2}{4}+b^2$$
$$= \tfrac{1}{2}(l_x^2+l_y^2) - \frac{r^2}{4} = \tfrac{1}{2}(l_x^2+l_y^2) - \frac{d(x,y)^2}{4}$$

So back in X we have



$$d(m,a)^2 \leq \tfrac{1}{2}\left(d(x,a)^2+d(a,y)^2\right) - \frac{d(x,y)^2}{4}$$

or

$$d(x,y)^2 \leq 2\left(d(x,a)^2+d(y,a)^2-2d(m,a)^2\right)$$

For a with $d(m,a)$ maximal, ie $d(m,a)=r(m,A)$

we got
$$d(x,y)^2 \leq 2\left(d(x,a)^2+d(y,a)^2-2r(m,A)^2\right)$$
$$\leq 2\left(r(x,A)^2+r(y,A)^2-2r(m,A)^2\right)$$

Since $r(m,A) \geq r(A)$ this becomes

$$d(x,y)^2 \leq 2\left(r(x,A)^2+r(y,A)^2-2r(A)^2\right) \checkmark$$

Now we can prove the theorem.

<u>Theorem</u> A CAT(0) group G has only finitely many conjugacy classes of finite subgroups.

<u>Proof</u>

Suppose H is a finite subgroup of a CAT(0) group G.

G acts properly and cocompactly by isometries on a CAT(0) space X, so H fixes the circumcenter x of an orbit, ie H is contained in the (finite) stabilizer G_x.

Cover X by translates of a compact set K containing x.
The action is proper, so $\Sigma = \{g \mid gK \cap K \neq \emptyset\}$ is finite. Since $G_x \subset \Sigma$, there are only finitely many possibilities for G_x, so for H.

If H fixes some $y \notin K$ then $y \in gK$ for some g, ie, $g^{-1}y = x \in K$ and $G_y = g G_x g^{-1}$ is conjugate to (one of the finitely many) G_x. for $x \in K$.

CAT(0) groups, like hyperbolic groups, are always finitely presented, but again the proof is very different.

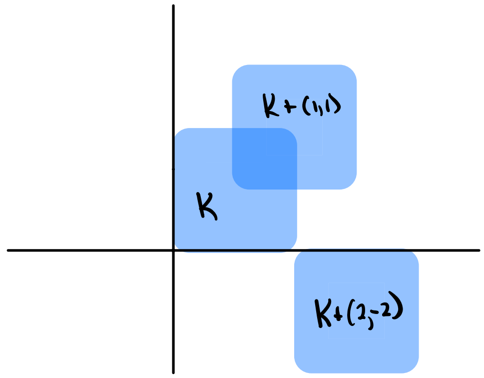
Hyperbolic groups are finitely-generated by definition, and we used short loops in (any) Cayley graph to get a set of relations.

For CAT(0) groups, it follows from the the Svarc-Milnor lemma that they are finitely generated

Recall the proof: $G \curvearrowright X$ CAT(0) proper, cocompact. Take $K \subset X$ compact such that $G \cdot U = X$ where U = interior(K).

Then $S = \{g \mid gU \cap U \neq \emptyset\}$ generates G, and S is finite because $S \subset \{g \mid gK \cap K \neq \emptyset\}$, which is finite because the action is proper

A simple example to think about is the usual picture of $G = \mathbb{Z}^2$ acting on $\mathbb{R}^2$. Take $K = [0,2] \times [0,2]$. Then translates of its interior $(0,2) \times (0,2)$ cover $\mathbb{R}^2$



$S = \{ (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1) \}$

We can eliminate inverses and just write

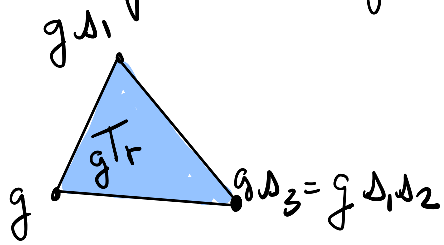
$S = \{ (1,0), (0,1), (1,1), (1,-1) \}$

Note the elements of S are not independent: $(1,0)+(0,1)=(1,1)$

<u>Theorem</u> Let S be the generators given by the S-M lemma, and let $R = \{ s_1 s_2 s_3^{-1} \mid s_i \in S,\ s_1 s_2 = s_3 \text{ and } U \cap s_1 U \cap s_3 U \neq \emptyset \}$

Then $G \cong \langle S \mid R \rangle$, ie $\ker F(S) \longrightarrow G$ is normally generated by R.

<u>Proof</u>: Look at $\mathcal{C}(G,S)$. For each $g \in G$ and each $r = s_1 s_2 s_3^{-1} \in R$, glue a triangle gT_r to the loop



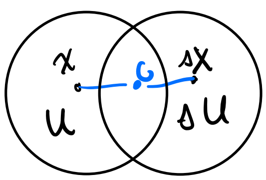
to get a complex $K := \mathcal{C} \cup \{ gT_r \mid g \in G, r \in R \}$

G acts freely and properly on K, so $K \to K/G$ is a covering space. If we can prove K is simply-connected, then K is the universal cover, and

$$G = \pi_1(K/G) = F(S)/\langle\langle R \rangle\rangle$$ by Van Kampen's theorem.

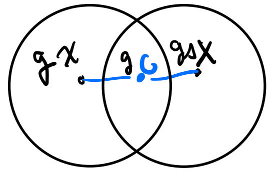
We do know X is simply connected; we will use this to show K is. First define a map $p : K \longrightarrow X$ as follows: choose $x \in X$.
Vertices of K are group elements g.
Define $p(g) = gx$.

There is an edge $g \bullet\!\!-\!\!-\!\!\bullet\, gs$ in K
$\iff U \cap sU \neq \emptyset$. Choose $c \in U \cap sU$.
We may assume U is connected (otherwise replace K by a ball containing it)



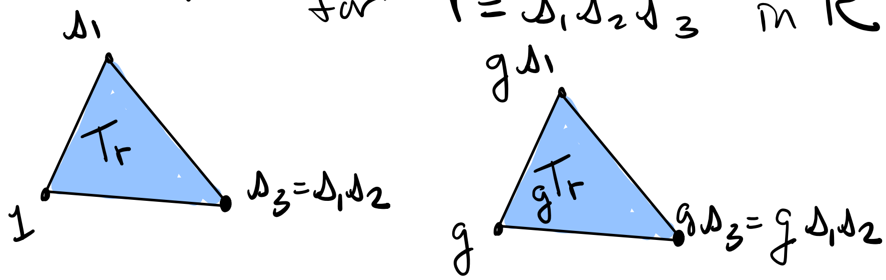
Connect x to c and c to sx

Then send $g \bullet\!\!-\!\!-\!\!\bullet\, gs$ to the path $gx - gc - gsx$:

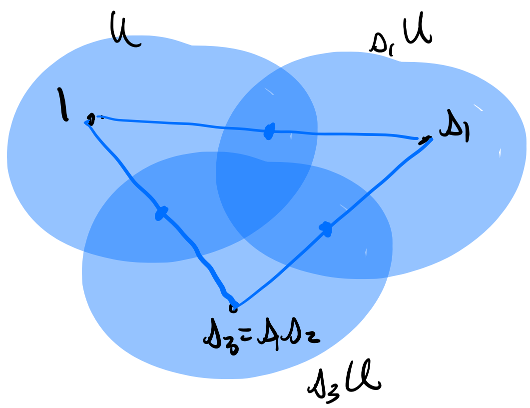


Now have $p : \mathcal{C}(G,S) \to X$.

Triangles in K are translates of T_r, for $r = s_1 s_2 s_3^{-1}$ in R



The image of ∂T_r in X is



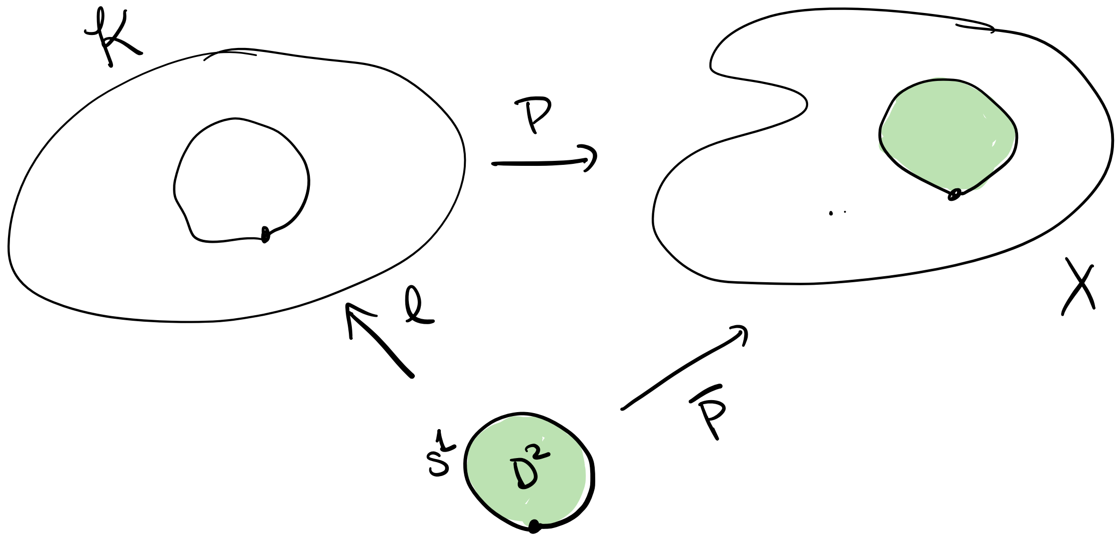
where $U \cap s_1 U \cap s_3 U \neq \emptyset$.

X is 1-connected, so $p|_{\partial T_r}$ extends to T_r;

Extend equivariantly to all triangles gT_r.

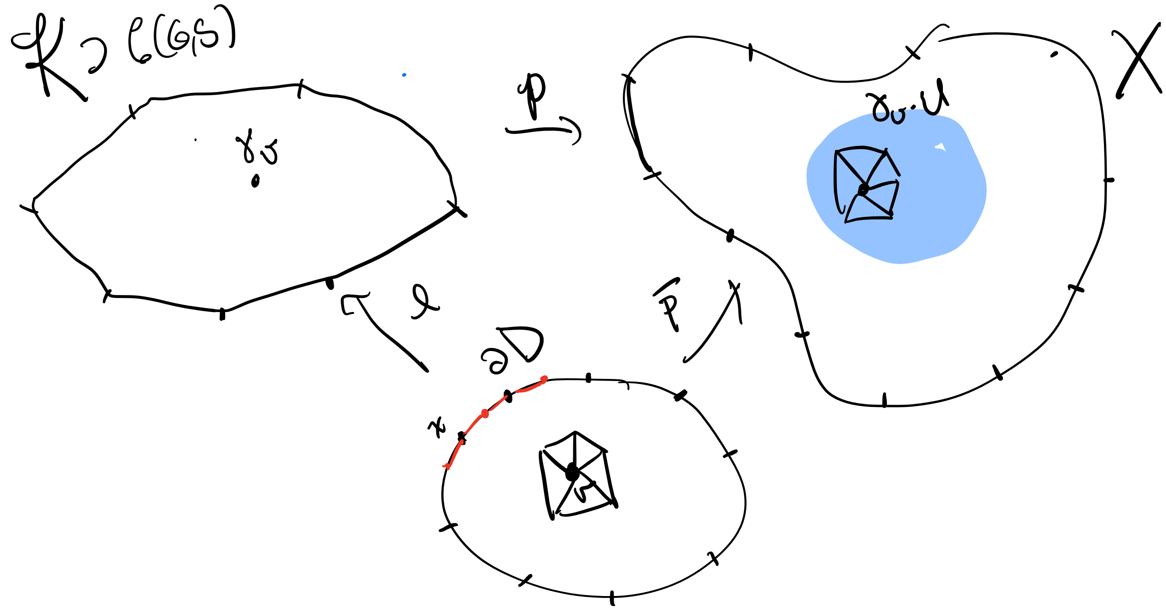
To show K is 1-connected: Let $\ell : S^1 \to K$ be a loop.

$p \circ \ell : S^1 \to X$ extends to $\bar{p} : D^2 \to X$ since X is 1-connected.

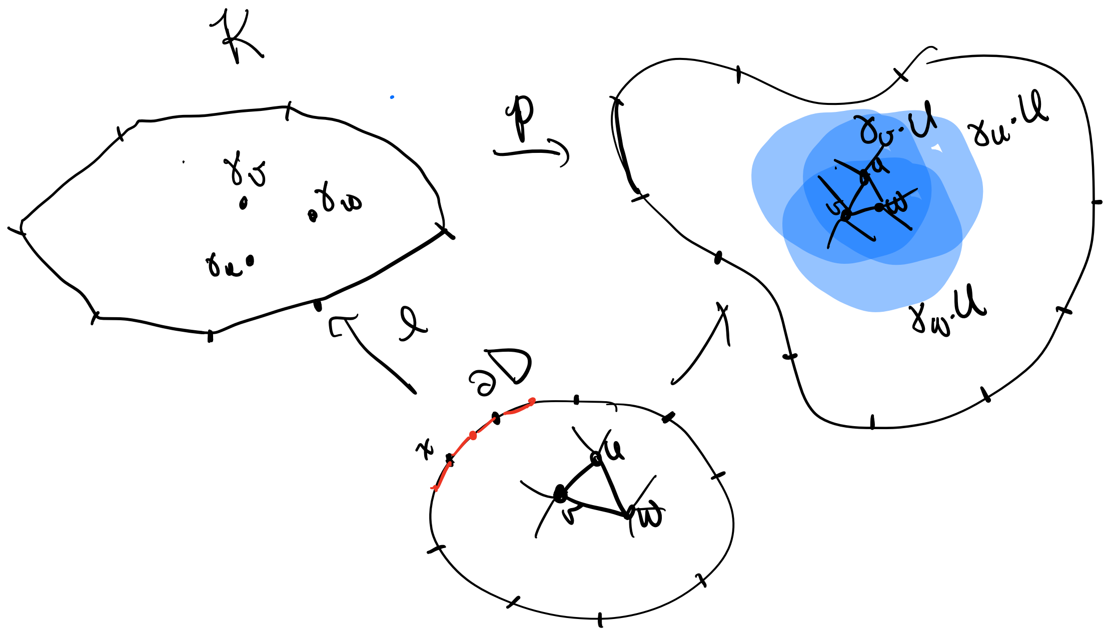


Triangulate D finely enough so that the all triangles around a vertex v map into a single translate $\gamma_v U_0$ (The Lebesgue lemma guarantees that you can do this: Let λ be the Lebesgue number of the cover $\{\bar{p}^{-1}(gU)\}$ of D^2, and subdivide until the triangles around v have diameter less than λ.)

Locate γ_v in $\mathcal{C}(G,S) \subseteq K$



Let $t = \Delta(u,v,w)$ be one triangle in D



Then $\Delta(uvw) \subset \gamma_u \cdot \mathcal{U} \cap \gamma_v \cdot \mathcal{U} \cap \gamma_w \cdot \mathcal{U}$

In particular, this intersection is not $\emptyset$.

$\gamma_v U \cap \gamma_u U \cap \gamma_w U$

$= \gamma_v \left(U \cap \gamma_v^{-1}\gamma_u U \cap \gamma_v^{-1}\gamma_w U \right) \neq \emptyset$

so $s_1 = \gamma_v^{-1}\gamma_u \in S$

$s_3 = \gamma_v^{-1}\gamma_w \in S$

and $s_2 = \gamma_u^{-1}\gamma_v \in S$ $\quad (\gamma_u^{-1}\gamma_v U \cap U \cap \gamma_u^{-1}\gamma_w U) \neq \emptyset$

Since $s_1 s_2 = s_3$, $s_1 s_2 s_3^{-1} \in R$

so there's a triangle in K with vertices $\gamma_u, \gamma_v, \gamma_w$

Now map $\Delta(u,v,w) \longrightarrow \Delta(\gamma_u \gamma_v \gamma_w) \subset K$.

Doing this for all triangles in D

extends $\ell : \partial D \longrightarrow K$

to $f : D \longrightarrow K$

$\therefore$ K is 1-connected and we're done

Below I list some other known properties of CAT(0) groups, and compare them to properties of hyperbolic groups!

We showed an infinite hyperbolic group has an infinite-order element.

* This is also true for CAT(0) groups. For hyperbolic groups we used the fact that they have only finitely many cone types, but that may not be true for CAT(0) groups (at least our proof doesn't work...)

If G is hyperbolic, we found a presentation with a linear Dehn function — and stated the fact that the Dehn function of <u>any</u> presentation is linear, and this characterizes hyperbolic groups.

* The Dehn function of a CAT(0) group is at most quadratic — but not all groups with quadratic Dehn functions are CAT(0)

* CAT(0) groups have solvable word and conjugacy problems

Hyperbolic groups satisfy a Tits alternative
every subgroup is either virtually cyclic or contains a copy of F_2

* Solvable subgroups of CAT(0) groups are virtually abelian

* Abelian subgroups of CAT(0) groups are finitely-generated

So a natural analog for CAT(0) groups would be

A subgroup of a CAT(0) group is either virtually abelian or contains a copy of F_2.

This is an open problem, though it is known to be true for some classes of CAT(0) groups.

Gromov's link condition

We've noted that CAT(0) groups have nice properties, but it is not so clear how to decide whether a group is CAT(0).

The best source of examples is the Cartan-Hadamard theorem, saying that if Y is NPC (locally CAT(0)) then its universal cover is CAT(0); since $\pi_1 Y$ acts properly (in fact freely) on $\tilde{Y}$, we get $\pi_1 Y$ is CAT(0) for compact NPC spaces Y.

But it's not so easy to decide whether a space is locally CAT(0)

If Y is a locally finite polyhedral complex, Gromov gave a necessary and sufficient condition.

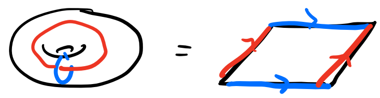
<u>Def</u> A <u>polyhedral complex</u> is a union of convex Euclidean polyhedra, glued together along faces by isometries

The metric is given by

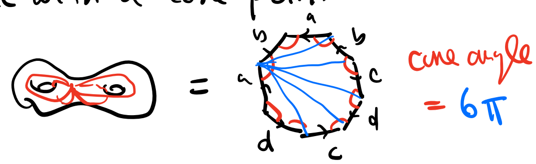
$$d(x,y) = \inf(\text{length of shortest piecewise linear path from } x \text{ to } y)$$

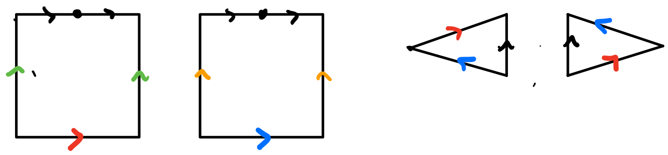
Examples

① A graph with positive edge lengths

② A flat torus 

③ A surface with a cone point



④ 

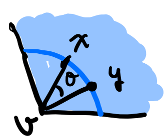
Gromov's link condition

Let Y be a polyhedral complex and v a vertex. Let s be the length of the shortest 1-cell in Y and $\varepsilon << s$. The link of v is

$$Lk(v) = \{ x \in X \mid d(x,v) = \varepsilon \}$$

The intersection of $Lk(v)$ with each n-cell is an $(n-1)$-cell contained in a Euclidean sphere of radius ε. Measure the distance between two points in this cell by the angle they form with v

$$d(x,y) = \angle xvy$$



$d(x,y) = \theta$

Theorem (Gromov) A polyhedral complex Y is NPC if and only if the link of every vertex is CAT(1)

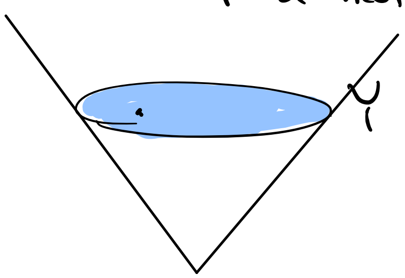
CAT(1) : Every triangle <u>with perimeter $< 2\pi$</u> is thinner than a triangle with the same edge lengths on the unit sphere $S^2 \subset \mathbb{R}^3$.

Comments on proof:

Points in the interior of top-dimensional cells have Euclidean ($\therefore$ CAT(0)) neighborhoods.

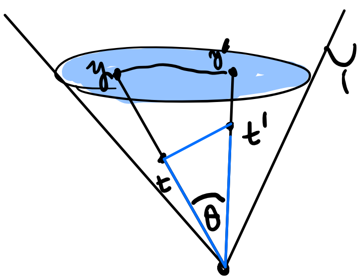
A neighborhood of a vertex v is a <u>cone</u> on $Lk(v)$

Y a metric space, $c(Y)$ its Euclidean cone



Topologically, $c(Y) = Y \times [0,\infty) / Y \times \{0\}$

The metric on $c(Y)$ is given as follows:

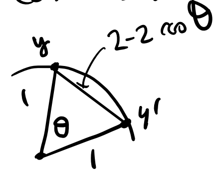


Let $d_\pi(y,y') = \min(\pi, d(y,y'))$. Then

$$d((y,t),(y',t'))^2 = t^2 + t'^2 - 2tt'(\cos d_\pi(y,y'))$$

If $Y \subset$ unit sphere in $\mathbb{R}^n$,
$d(y,y') =$ angle $\angle y o y'$, so this is the law of cosines:

$$t = t' = 1 \Rightarrow d = \begin{cases} 2 & \text{if } d(y,y') \geq \pi \\ 2 - 2\cos d_Y(y,y') & \text{if } d(y,y') < \pi \end{cases}$$



<u>Berestovskii's Theorem</u>: $c(Y)$ is CAT(0) if and only if Y is CAT(1).

<u>Pf</u> Bridson-Haefliger Thm II.3.14 (p.188)

So this shows that vertices have CAT(0) neighborhoods.

A neighborhood of a point on an edge adjacent to v is $(-\varepsilon,\varepsilon) \times c(Lk_L(u))$ where $L = Lk_Y(v)$.

Berestovskii's theorem also works for CAT(k) spaces, in particular, one can define a spherical cone $C_1(Y)$ and prove

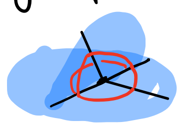
$C_1(Y)$ is CAT(1) if and only if Y is CAT(1).

So this will give a CAT(0) neighborhood of points on edges.

Points on faces of larger dimension are also of the form $B_\varepsilon \times c(Lk_L(u))$, where B_ε is a Euclidean ball and L is a link in a previous link, and the theorem is proved by induction. //

Even with Gromov's theorem, it's hard to tell whether a polyhedral complex is CAT(0) - unless it's 2-dimensional

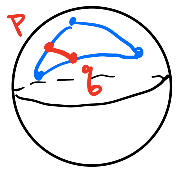
If Y is 2-dimensional, the link of every vertex is (at most) 1-dimensional, ie it's a graph plus some isolated points



When is a graph CAT(1)? We just have to check triangles with perimeter $< 2\pi$

in S^2, such a triangle lives in a hemisphere

If you have a triangle in a graph, the only way to get to another side is to travel through the edges. But on the sphere there's a shorter path:



Ie a graph is cat(1) iff it has no loops of length $< 2\pi$.

Example ④ above of a polyhedral complex satisfies this criterion (exercice!), so the fundamental group G is CAT(0). By van Kampen's theorem G has the presentation

$$G = \langle a, b, s, t \mid [a,b]=1,\ s^{-1}(ab)^2 s = b,\ t^{-1}(ab)^2 t = a \rangle$$

The map $G \to G$ sending $a \to a^2$, $b \to b^2$, $s \to s$ and $t \to t$ sends the relations to id, so induces a homomorphism
$$f : G \to G$$

It's easy to see f is surjective.
Dani Wise showed f is not injective.

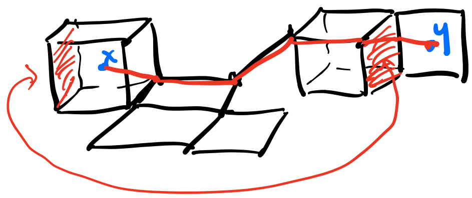
For finitely generated linear groups (ie subgroups of $GL(n,\mathbb{C})$), any surjective homomorphism is an isomorphism, so this proves that G is not linear

This was the first example of a CAT(0) group which is not linear, answering a question of Gromov.

If Y is more than two-dimensional, Gromov's link condition becomes difficult to check — <u>unless</u> all the cells are cubes $[0,1]^n$.

<u>Cube complexes</u> - Polygonal complexes, all polygons are cubes $[0,1]^n$

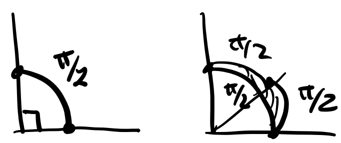
<u>Picture</u>: metric cubes $[0,1]^n$ glued together by isometries of their faces.



$d(x,y)$ = length of shortest path from x to y.

It turns out there is a purely combinatorial way to check the link condition for cube complexes:

In a cube, the link of a vertex is an "all-right" spherical simplex:



So $Lk(v)$ is a simplicial complex, and all edges have length $\pi/2$.

Note that the boundary of a 3-dim'l cube is not locally CAT(0): at a corner we have a triangle that is fatter than a Euclidean comparison triangle:

$d(p,q) > d(\bar{p},\bar{q})$

But if you fill in the cube, this problem disappears.

Def A simplicial complex Σ is a flag complex if whenever Σ contains the 1-skeleton of a simplex, it also contains the simplex

Theorem (Gromov) A cube complex is NPC if and only if the link of every vertex is a flag complex

Proof By Gromov's theorem,

It suffices to prove that a simplicial complex made of all-right spherical simplices is CAT(1) if and only if it is flag. This is proved by induction on dim(Y).

If dim(Y) = 2, this says that a link graph cannot contain a triangle with all side lengths $\pi/2$, so is a direct consequence of Gromov's link condition, as we discussed above

If dim Y > 2, this requires some spherical geometry plus induction.

For the full proof, see Bridson-Haefliger, theorem 5.18. //

Cube complexes and RAAGs

We have:

Theorem (Gromov) A locally finite cube complex Y is NPC if and only if the link of every vertex is a flag complex

We can use this to prove certain groups are CAT(0):

<u>Definition</u> A right-angled Artin group (RAAG) is a finitely-generated group with a presentation $\langle S \mid R \rangle$, where all $r \in R$ are of the form $s_1 s_2 s_1^{-1} s_2^{-1}$ (ie $s_1 s_2 = s_2 s_1$) for $s_1, s_2 \in S$.

The presentation is conveniently encoded by drawing a (simplicial) graph Γ

vertices = elements of S

edge $s_1 — s_2 \iff s_1 s_2 s_1^{-1} s_2^{-1} \in R$

The RAAG is then called A_Γ

eg $\Gamma =$ 

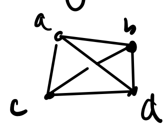
$$A_\Gamma = \langle a, b, c \mid ab = ba \rangle = \underset{\langle a,b\rangle}{\mathbb{Z}^2} * \underset{\langle c \rangle}{\mathbb{Z}}$$

RAAGs include free groups:

$\Gamma =$ 

(no edges)

and free abelian groups:

$\Gamma =$ 

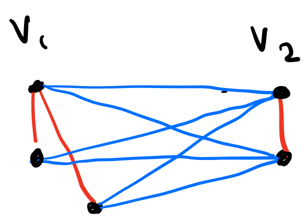
(complete graph)

Free products of RAAGs are RAAGs:

If $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $A_\Gamma = A_{\Gamma_1} * A_{\Gamma_2}$

Direct products of RAAGs are RAAGs:

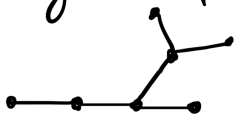
Γ is a <u>join</u> if $V(\Gamma) = V_1 \sqcup V_2$, and every vertex of V_1 is connected to every every vertex of V_2:



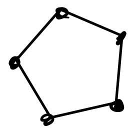
Then $A_\Gamma = A_{\Gamma_1} \times A_{\Gamma_2}$

But in general there is no simple way to decompose A_Γ

Interesting examples of RAAGs include

$\Gamma =$  $= \pi_1(\text{3-manifold})$ (True whenever Γ is a tree)

RAAGs contain surprising subgroups

$\Gamma =$  $\supseteq \pi_1(\Sigma_g)$ (surface group)

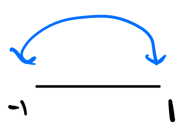
<u>Claim</u> A_Γ is always CAT(0)

<u>Proof</u>: We exhibit A_Γ as π_1 of an NPC cube complex S_Γ.

First, recall that a k-clique in Γ is a set of k vertices $v_1, \dots, v_k$ that form a complete subgraph.

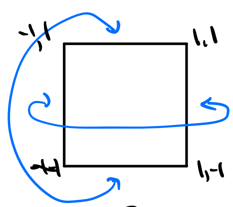
We give two descriptions of S_Γ

① Let T^n be the n-Torus $[-1,1]^n/\text{opposite sides}$:



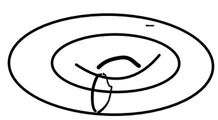
T^1

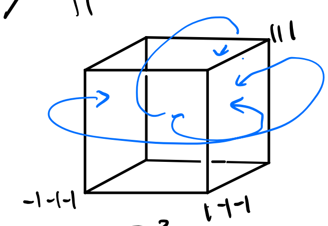
$\parallel$



T^2

$\parallel$

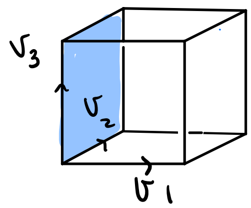




T^3

(can't draw)

k-dimensional faces of $[-1,1]^n$ at $(-1,\dots,-1)$
$\leftrightarrow$ subsets of $\{1,\dots n\}$

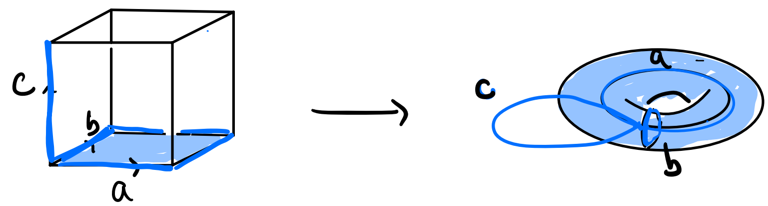


$\leftrightarrow \{2,3\}$

If Γ has n vertices, let $K_\Gamma \subset [-1,1]^n$ be the subcomplex consisting of faces corresponding to cliques in Γ, and S_Γ the image of of K_Γ in T^n:

eg $\Gamma =$

$K_\Gamma =$



$= S_\Gamma$

② Second description of S_Γ

S_Γ is the cube complex formed as follows.

- start with one 0-cube (vertex) p
- attach an edge for each vertex of Γ (= generator of A_Γ) – it becomes a loop
- Attach a square for each edge of Γ – it becomes a torus

So far we have the presentation complex for A_Γ, ie a 2-dimensional complex with $\pi_1 = A_\Gamma$ (by Van Kampen)

But we've not done

- Attach a k-cube for each k-clique in Γ - it becomes a k-torus

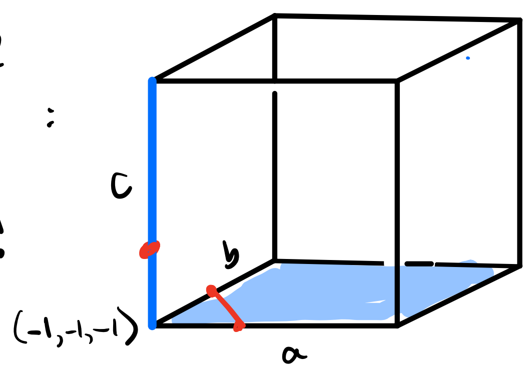
Now we're done.

Attaching higher-dimensional cubes does not change π_1 ('Any loop in X can be pushed into the 2-skeleton'), so $\pi_1(S_\Gamma) = A_\Gamma$.

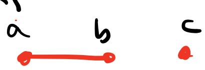
<u>Claim</u>: S_Γ is an NPC complex.

We use Gromov's link condition for cube complexes. There's only one link to check since there's only one vertex

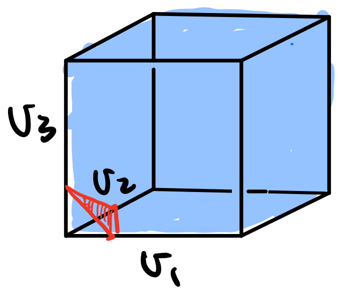
In $[-1,1]^n$ we have a subcomplex K_Γ:



$lk_{K_\Gamma}(-1,-1,-1) = \Gamma$!



If $v_1, \dots, v_k$ are a clique then every pair $v_i v_j$ is joined by an edge, so we have the boundary of a $(k-1)$-simplex. But we've filled in the cube, so this $(k-1)$-simplex bounds a simplex in $\mathrm{lk}(-1, \dots, -1)$



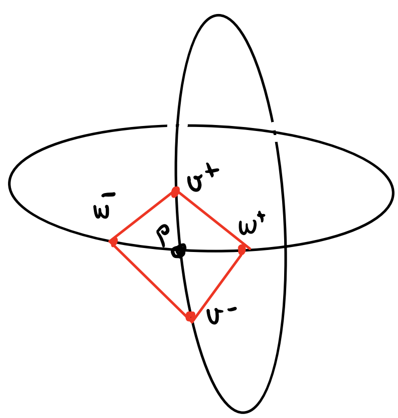
<u>Definition</u> The <u>flag completion</u> $\tilde{\Gamma}$ is the flag complex with 1-skeleton Γ.

So $\mathrm{lk}_{K_\Gamma}(-1, \dots, -1) = \tilde{\Gamma}$.

But this is the link in K_Γ, not in S_Γ.

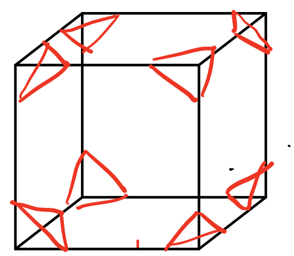
In T^n each edge becomes a loop at p. each vertex in Γ corresponds to 2 vertices v^+, v^- in $\mathrm{lk}_{S_\Gamma}(p)$:

$v^\pm, w^\pm$ are joined by an edge $\Leftrightarrow$ v, w are joined by an edge in Γ :



<u>Definition</u> The double $D(\Gamma)$ of a simplicial graph Γ is the graph with two vertices v^+, v^- for every vertex of Γ and an edge $v^\pm$ to $w^\mp$ for every edge v to w in Γ.

Vertices $v_1^\pm \dots v_k^\pm$ form a clique in $D(\Gamma)$ and only if $v_1 \dots v_k$ forms a clique in Γ, so that $v_1 \dots v_k$ are the vertices of a $(k-1)$-cube $C \subset K_\Gamma$



All 2^k cliques are filled in, by the images of the links of the 2^k vertices of C in S_Γ.

ie The link $lk_{S_\Gamma}(p)$ is the flag completion $\widetilde{D(\Gamma)}$ of $D(\Gamma)$.

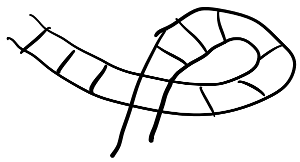
Since this is flag, S_Γ is NPC and $\pi_1 S_\Gamma = A_\Gamma$ is CAT(0).

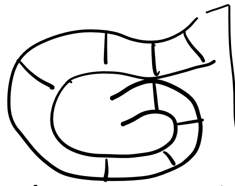
Note that Γ is an arbitrary simplicial graph so S_Γ can be pretty wild...

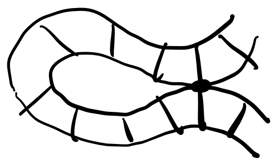
but not <u>too</u> wild:

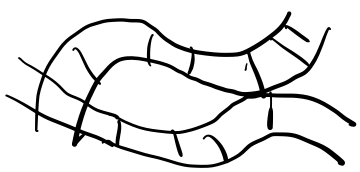
They don't contain:

① Mobius bands 

②  self-intersecting cube chains


(indirect is OK)

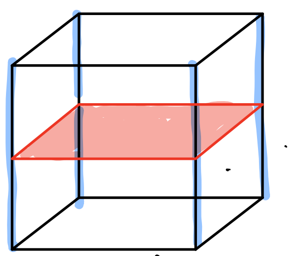
③  direct self-osculating chains

④  inter-osculating chains

<u>Def</u> a cube complex satisfying ①-④ is a <u>special</u> cube complex

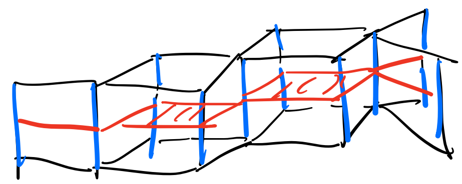
Not all special cube complexes are Salvetti complexes S_Γ, but Haglund and Wise showed they are closely related, by studying <u>hyperplanes</u> in special cube complexes.

Def The <u>midplane</u> of a cube $C=[-1,1]^n$ is the intersection of a coordinate hyperplane with C



The midplanes give an $\sim$ relation on the edges of a cube

The equivalence relation on edges of X generated by a single edge gives a <u>hyperplane</u> in X



Given a special cube complex Y, define a graph (called the <u>crossing</u> graph of Y)

vertices = hyperplanes H

edge $H - H'$ if H and H' intersect transversely.

Thm (Haglund-Wise) If X is a special NPC cube complex, then $\pi_1 X \hookrightarrow A_\Gamma$, where Γ is the crossing graph of X.

One nice feature of RAAGs is the following:

<u>Theorem</u> (Davis-Januszkiewicz): RAAGs are linear

(Recall we had an example of a CAT(0) group that is <u>not</u> linear a couple of lectures ago)

The proof proceeds by proving every RAAG embeds as a finite index subgroup of a right-angled <u>Coxeter</u> group C_Γ

<u>Def</u>: Let Γ be a finite simplicial graph. The right-angled Coxeter group C_Γ is the group given by the presentation $\langle S \mid R \rangle$ where

- S = vertices of Γ
- $s^2 \in R$ for all $s \in S$
- All other elements of R are of the form $s_1 s_2 s_1^{-1} s_2^{-1}$

Right-angled Coxeter groups can be realised as linear reflection groups: the generators correspond to reflections in hyperplanes, they commute if the hyperplanes intersect at right angles

Major earthquake in 3-manifold theory:

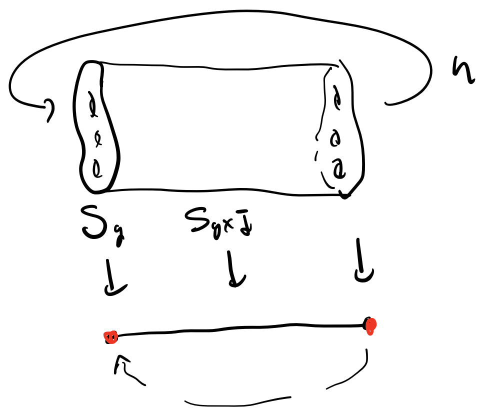
Thm (Agol, 2012): M^3 a compact orientable irreducible hyperbolic 3-manifold has a finite cover whose fundamental group acts properly and cocompactly on a special CAT(0) cube complex

ie $\pi_1 M^3$ has a finite index subgroup that embeds in a RAAG.

In particular, this subgp is linear ($\Rightarrow$ Hopfian, resid. finite, Tits, ...)

This was the last piece of a puzzle about the structure of 3-manifolds:

Every closed hyperbolic 3-manifold has a finite cover that is a surface bundle over a circle:



glue ends together by a homeomorphism h

$$M^3 = S_g \times [0,1] \Big/ (x,0) \sim (h(x),1)$$

(True for all closed, orientable, irreducible 3-manifolds, using Perelman + more work (2018))

Finally, a very quick word about automorphism groups $\mathrm{Aut}(G)$ and

$\mathrm{Out}(G) = \mathrm{Aut}(G) / \mathrm{Inn}(G)$ for G hyperbolic or CAT(0)

eg $\quad \mathrm{Aut}(\mathbb{Z}^n) = GL_n\mathbb{Z} = \mathrm{Out}(\mathbb{Z}^n)$

$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$

$\mathrm{Out}(A_\Gamma) = \mathrm{Aut}/\mathrm{Inn}$

$\mathrm{Out}(\pi_1 S_n) = \mathrm{Aut}/\mathrm{Inn}$

$= \Gamma_g$ mapping class grp

These groups are almost never hyperbolic or CAT(0) ($SL_2\mathbb{Z}$ is an exception)

$SL_2\mathbb{Z}$ acts on $\mathbb{H}$ properly but <u>not</u> cocompactly, so is not QI to $\mathbb{H}$

But you can still use this action and the geometry of $\mathbb{H}$ to study the group.

There are analogous spaces $\mathcal{O}_G$

$G = SL_n\mathbb{Z}$, $\mathrm{Out}(F_n)$, $\mathrm{Out}(\pi_1 S_g)$, $\mathrm{Out}(A_\Gamma)$

with proper actions that are not cocompact (though this can be finessed)

Points in $\mathcal{O}_G$ are (CAT(0), hyperbolic) spaces X together with an isomorphism $\mu: \pi_1(X) \xrightarrow{\cong} G$, called a <u>marking</u>

An automorphism of G acts by changing the marking

$$\begin{tikzcd} G & \pi_1 X \arrow[l, "\cong"', "\mu"] \\ G \arrow[u, "\alpha"] \arrow[ur, "\mu\circ\alpha"'] & \end{tikzcd}$$

, **<u>not</u>** the metric on X

You move around the space by deforming the metric on X (think about deforming the parallelogram that defines a flat torus)

You can study the topology and geometry of $\mathcal{O}_G$ to learn about $\mathrm{Out}(G)$.

This is classical for $G = \mathbb{Z}^n$ (Symmetric spaces)

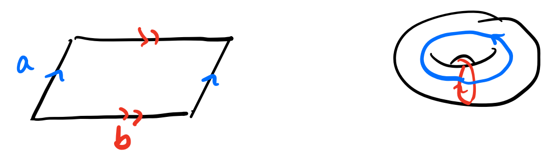
slightly newer for $G = \pi_1 S_g$ (Teichmuller space)

slightly newer for $G = F_n$ (Outer space)

very recent for $G = A_\Gamma$.

For $G = \mathbb{Z}^n$, use X = flat torus.

eg $n=2$



Different shaped parallelograms give different points in $\mathcal{O}_G$

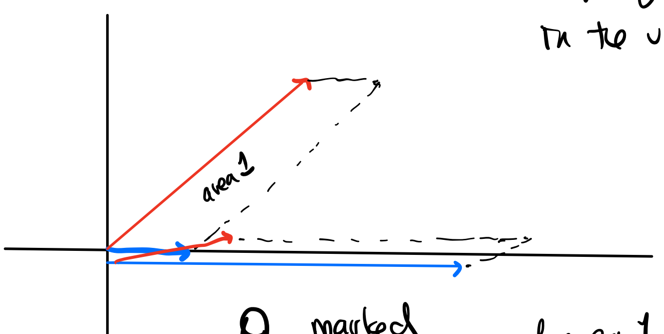
Some parallelograms give the same (ie isometric) flat torus:

eg

but with a different marking, ie choice of isomorphism

$$\mathbb{Z}^2 \longrightarrow \pi_1(T)$$

If we normalize so that area$(T^2)=1$ and put one edge on the x-axis, the other edge gives a unique point in the upper half-plane



So the space $\mathcal{O}_{\mathbb{Z}^n}$ of marked flat tori of area 1 can be identified with $\mathbb{H}^2$.

$\mathbb{H}^2 = SL(2,\mathbb{R})/SO(2)$ = symmetric space for GL_2

The action of $SL(2,\mathbb{Z})$ on $\mathcal{O}_{\mathbb{Z}^n}$ is the action by Möbius transformations that we already studied.

For $n>2$, also get the space of marked flat tori of volume 1 =

$$SL(n,\mathbb{R})/SO(n) \quad \left(= GL(n,\mathbb{R})/O(n)\right)$$

and $SL_n(\mathbb{Z})$ acts on the left by changing the marking but not the isometry type of the torus.

For $G = \pi_1 \Sigma_g$ we use hyperbolic surfaces of area 1 homeomorphic to Σ_g

$\mathcal{O}_{\pi_1 \Sigma_g}$ is called Teichmuller space

For $G = F_n$ we use metric graphs with total edge length 1

$\mathcal{O}_{F_n}$ is called "Outer space"

For $G = A_\Gamma$ we use special cube complexes with $\pi_1 \cong A_\Gamma$ (Salvettis, twisted Salvettis, (twisted) blowups of Salvettis) to get an Outer space