Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in Exercise 7.5 (via Moodle) by noon on Friday of week nine (2025-03-07).

In the following problems (X, d_X) is a proper geodesic metric space.

Exercise 7.1. Prove that X is complete.

Exercise 7.2. Suppose that $\Delta \subset X$ is finite and non-empty. Let $\mathrm{Cen}(\Delta)$ be the set of outcentres of Δ.

1. Prove that $\mathrm{Cen}(\Delta)$ is non-empty.
2. Suppose that h is an isometry of X. Prove that $h(\mathrm{Cen}(\Delta)) = \mathrm{Cen}(h(\Delta))$.

Now suppose that (G, S) is a finitely generated group. Let $\Gamma_S = \Gamma(G, S)$ be the resulting Cayley graph.

Exercise 7.3. Suppose that $F < G$ is a finite subgroup. Prove the following.

1. The action of F on Γ_S (by left multiplication) preserves $\mathrm{Cen}(F)$. Furthermore, the action of F on $\mathrm{Cen}(F)$ is *free*: if $f \in F$ fixes some $g \in \mathrm{Cen}(F)$ then $f = 1_G$.
2. Suppose that h lies in G. Then hFh^{-1} preserves, and acts freely on, $\mathrm{Cen}(hF)$.

Exercise 7.4. Suppose that X is δ–hyperbolic. Suppose that $k \geq 4\delta$. Suppose that $n > 0$. Suppose that $\alpha\colon [0, n] \to X$ is a k–local geodesic. Prove that

$$\frac{n}{k/2} \leq d_X(\alpha(0), \alpha(n)) \leq n$$

Exercise 7.5. Suppose that

$$G = \langle S \mid R \rangle = \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$$

Prove that this is a Dehn presentation. [This was only sketched in Lecture 20 – provide the details.]

Exercise 7.6. Suppose that $G = \langle S \mid R \rangle$ is a Dehn presentation. Give a linear-time algorithm for the word problem in G.