Please let me (Saul) know if any of the problems are unclear or have typos. There are no problems to turn in from this exercise sheet.

Exercise 8.1. Suppose that (G, S) is a δ–hyperbolic group. Prove that there is a constant $K = K(|S|, \delta)$ with the following property. Suppose that u and v are conjugate elements of G. Then there is an element $w \in G$ so that

1. $uw = wv$ and
2. $|w|_S \leq |u|_S + |v|_S + K$.

Exercise 8.2. Suppose that (G, S) is δ–hyperbolic. Prove that there exists an algorithm to solve the conjugacy problem in G.

Exercise 8.3. Suppose that (G, S) is a finitely generated group. Suppose that $H < G$ is a quasi-convex subgroup. Prove that

1. H is finitely generated and
2. H is undistorted in G.

[I sketched this in Lecture 22 and provided more details in the notes. However, you should work through the proof, especially of the second part.]

Exercise 8.4. [Harder.] Suppose that (G, S) is a finitely generated group. Suppose that $H, K < G$ are quasi-convex subgroups. Prove that $H \cap K < G$ is also a quasi-convex subgroup.

Exercise 8.5. Suppose that (G, S) is a δ–hyperbolic group. Suppose that $H < G$ is a subgroup. Suppose that H is finitely generated and undistorted. Prove that H is quasi-convex. [This will be easier once we have finished the material in week nine.]

Exercise 8.6. Suppose that (G, S) is a δ–hyperbolic group. Suppose that $H < G$ is a quasi-convex subgroup. Prove that H is δ'–hyperbolic for some δ'. [This will be easier once we have finished the material in week nine.]

Exercise 8.7. Suppose that G is a group. Suppose that $g \in G$ is an element. Define the *centraliser* of g to be the subset

$$C_G(g) = \{h \in G \mid gh = hg\}$$

Prove that the centraliser is a subgroup of G. Prove that g is central in $C_G(g)$.

Exercise 8.8. Suppose that (G, S) is δ–hyperbolic. Suppose that $g, h \in G$ are elements with $h \in C_G(g)$. Suppose that $p \in [1, h]$ and $q \in g \cdot [1, h]$ have $d_S(p, q) \leq 2\delta$. Prove that $d_S(p, gp) \leq 2\delta + |g|_S$.

Exercise 8.9. Suppose that (X, d_X) is δ–hyperbolic. Fix $R, \epsilon \geq 0$. Fix $x, y, z \in X$ so that

$$d_X(x, y) \geq R \qquad d_X(x, z) \geq R \qquad d_X(y, z) \leq \epsilon$$

Fix $p \in [x, y]$ and $q \in [x, z]$ so that

$$d_X(x, p) = d_X(x, q) = R$$

Prove that $d_X(p, q) \leq 2\delta + \epsilon$.

Exercise 8.10. Suppose that (G, S) is δ–hyperbolic, infinite, and not virtually $\mathbb{Z}$. Prove that G has exponential growth. [I sketched the case of $e(G) = 1$ in Lecture 24.]