Please let me (Saul) know if any of the problems are unclear or have typos. There are no problems to turn in from this exercise sheet.

Exercise 9.1. Suppose that (X, d_X) is a δ–hyperbolic metric space. Suppose that $\alpha\colon [a,b] \to X$ is a (λ, C)–quasi-geodesic. In an abuse of notation, we also use α to denote the image of α. Suppose that $\beta = [\alpha(a), \alpha(b)]$ is a geodesic. We proved in Lecture 25 that there is a constant $D = D(\delta, \lambda, C)$ so that α is contained in $N_X(\beta, D)$, the D–neighbourhood of β. Prove that there is a constant $D' = D'(\delta, \lambda, C)$ so that β is contained in $N_X(\alpha, D')$.

Exercise 9.2. Suppose that X, Y, and Z are metric spaces. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are quasi-isometric embeddings. Prove that $g \circ f$ is a quasi-isometric embedding (perhaps with worse constants).

Exercise 9.3. Suppose that X and Y are metric spaces. We write $X \overset{\text{QI}}{\simeq} Y$ if X and Y are quasi-isometric (for some constants λ, C, and K). Prove that the relation $\overset{\text{QI}}{\simeq}$ is an equivalence relation on the class of metric spaces.

Exercise 9.4. Suppose that n is a natural number. Suppose that $1 \leq p, q \leq \infty$ are real numbers. Prove that the identity map from $(\mathbb{R}^n, \ell^p)$ to $(\mathbb{R}^n, \ell^q)$ is a quasi-isometry.

Exercise 9.5. Suppose that X, X', Y, and Y' are metric spaces. Suppose that $X \overset{\text{QI}}{\simeq} X'$ and $Y \overset{\text{QI}}{\simeq} Y'$. Prove that $X \times Y \overset{\text{QI}}{\simeq} X' \times Y'$.

Exercise 9.6. Prove, directly or using the above, that $\mathbb{Z}^n \overset{\text{QI}}{\simeq} \mathbb{R}^n$.

Exercise 9.7. Suppose that G and H are groups. Suppose that S and T are finite generating sets for G and H, respectively. Suppose that $\Gamma(G, S)$ is quasi-isometric to $\Gamma(H, T)$. Prove the following.

1. $e(G) = e(H)$.
2. G and H have equivalent growth rates.
3. If G is virtually nilpotent then so is H.
4. If G is virtually abelian then so is H. [Harder.]
5. If G is finitely presented then so is H. [Harder.]

Exercise 9.8. Suppose that X is a metric space. Suppose that G is a group, given with a geometric action on X (as defined in Lecture 26). Let X' be the quotient $G\backslash X$. We define $d_{X'}\colon X' \times X' \to \mathbb{R}$ by

$$d_{X'}([x], [y]) = \inf\{d_X(gx, hy) \mid g, h \in G\}$$

1. Prove that the infimum in the definition is realised.
2. Prove that $(X', d_{X'})$ is a geodesic metric space.
3. Prove that the metric topology on X' agrees with the quotient topology.
4. Deduce that X' has bounded diameter.

Exercise 9.9. Suppose that G is a group and $H < G$ is a finite index subgroup. Suppose that S and T are finite generating sets for G and H, respectively.

1. Prove that the inclusion of H into G induces a quasi-isometry from $\Gamma(H, T)$ to $\Gamma(G, S)$.
2. Prove that all finitely generated nonabelian free groups are quasi-isometric.
3. Suppose that S_g is the closed, oriented, connected surface of genus g. Prove that, for all $g, h \geq 2$, the fundamental group $\pi_1(S_g)$ is quasi-isometric to $\pi_1(S_h)$.

Exercise 9.10. Suppose that $H < G$ is a subgroup. Suppose that T and S are finite generating sets for H and G, respectively. Prove that the following are equivalent.

1. H is undistorted in G.
2. H is quasi-convex in G.
3. Inclusion of H into G induces a quasi-isometric embedding.

Exercise 9.11. Suppose that (G, S) is δ–hyperbolic. Suppose that $g \in G$ is non-torsion. Let $C_G(g)$ be the centraliser of g. Suppose that T is a finite generating set for $C_G(g)$. Prove that $Z(C_G(g))$, the centre of the centraliser, equals the intersection $\cap_{t \in T} C_G(t)$. Deduce that $Z(C_G(g))$ is quasi-convex in G, so is finitely generated, and thus is undistorted in G.

Exercise 9.12. Suppose that G is finitely generated by S. For any g in G we define

$$\tau_S(g) = \lim_{n \to \infty} \frac{1}{n} d_S(1_G, g^n)$$

1. Prove that $\tau_S(g)$ is well-defined.
2. Prove that $\tau_S(g^m) = |m| \cdot \tau_S(g)$ for all $m \in \mathbb{Z}$.
3. Prove that $\tau_S(g) = \tau_S(h)$ for any $h \in G$ conjugate to g.
4. Suppose that (G, S) is δ–hyperbolic. Prove that $\tau_S(g) = 0$ if and only if g has finite order.