

Please post on the Q&A forum if any of the problems have mistakes, or are unclear, or can be improved. Thank you!

Exercise 1.1. We define the real-valued trigonometric functions sine and cosine by their power series.

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

From this, prove the following.

1. Both series converge for all real x .
2. $\cos(0) = 1$ and $\sin(0) = 0$.
3. Cosine is even; sine is odd.
4. The derivative of cosine is negative sine; the derivative of sine is cosine.
5. The addition laws:

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \cos(x)\sin(y) + \sin(x)\cos(y)\end{aligned}$$

6. The circle law: $\cos^2(x) + \sin^2(x) = 1$.
7. Prove that cosine has a smallest positive root. Denote this by x_0 . Prove that

$$1.4 < \sqrt{2} < x_0 < \sqrt{6 - 2\sqrt{3}} < 1.6$$

[Challenge: In fact x_0 is *much* closer to the upper bound. Why? Further challenge: Find more decimal places, directly from the definitions.]

8. Prove that $\sin(x_0) = 1$.
9. Prove that

$$\cos(x+x_0) = -\sin(x) \quad \text{and} \quad \sin(x+x_0) = \cos(x)$$

10. Prove that

$$\cos(x+4x_0) = \cos(x) \quad \text{and} \quad \sin(x+4x_0) = \sin(x)$$

Thus $4x_0$ is a *period* of cosine (and of sine).

11. Prove that $4x_0$ is the smallest positive period of cosine (and of sine).

We now define $\pi = 2x_0$. Define $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ by $\gamma(\theta) = (\cos(\theta), \sin(\theta))$.

12. Prove that γ is a unit-speed parametrisation of the unit circle. Deduce from this (and the above) that γ is a simple contour. Deduce that the arc-length of the unit circle is 2π . $\diamond$

Exercise 1.2. Suppose that z and w are complex numbers. Prove $|z + w| \geq |z| - |w|$. $\diamond$

Exercise 1.3. Suppose that $C = C(0; 1)$ is the unit circle, oriented anticlockwise. Compute the contour integral $I(n) = \int_C (\bar{z})^n dz$ where n is an integer. $\diamond$

Exercise 1.4. We quote the entirety of [Thomas Clausen, Aufgabe 53, J. Reine Angew. Math. 2 (1827), 286–287].

Aufgabe. Wenn e die Basis der hyperbolischen Logarithmen, π den halben Kreisumfang, und n eine positive oder negative ganze Zahl bedeuten, so ist bekanntlich $e^{2n\pi\sqrt{-1}} = 1$, $e^{1+2n\pi\sqrt{-1}} = e$; folglich auch $e^{(1+2n\pi\sqrt{-1})^2} = e = e^{1+4n\pi\sqrt{-1}-4n^2\pi^2}$. Da aber $e^{1+4n\pi\sqrt{-1}} = e$ ist, so würde daraus folgen: $e^{-4n^2\pi^2} = 1$, welches absurd ist. Nachzuweisen, wo in der Herleitung dieses Resultats gefehlt ist.

Here is a very loose translation:

For any integer n we have $e^{2\pi in} = 1$. Multiplying both sides by e we obtain $e^{1+2\pi in} = e$. We now compute as follows.

$$\begin{aligned} e &= e^{1+2\pi in} \\ &= (e^{1+2\pi in})^{1+2\pi in} \\ &= e^{(1+2\pi in)^2} \\ &= e^{1+4\pi in-4\pi^2 n^2} \\ &= e^{1-4\pi^2 n^2} \end{aligned}$$

Dividing both sides by e we deduce that 1 tends to zero as n tends to infinity. This is absurd. So, find (and explain) the mistake. $\diamond$