Please post on the Q&A forum if any of the problems have mistakes, or are unclear, or can be improved. Thank you!

Exercise 2.1. Take $\omega = e^{\pi i/4}$. Define $E\colon \mathbb{C} \to \mathbb{C}$ to be the rotation $E(z) = \omega z$. Define $D\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by $D(z) = 2z/(1-z^2)$. Define $\gamma\colon \mathbb{R} \to \mathbb{C} \cup \{\infty\}$ by $\gamma(t) = D(E(t))$. Sketch the image of γ. In your sketch, include the real and imaginary axes, indicate the orientation of γ, and label points of interest (including the images of the integers and "infinity"). [This exercise was inspired by Figure 8 of the article *Reflections on the lemniscate of Bernoulli: the forty-eight faces of a mathematical gem* by Langer and Singer.] ◇

Exercise 2.2. Suppose that $\gamma\colon [0, 2\pi] \to \mathbb{C}$ is the usual parametrisation of the unit circle: $\gamma(\theta) = e^{i\theta}$. For each function f below, sketch the path $f \circ \gamma$. In your sketch, include the real and imaginary axes, indicate the orientation of $f \circ \gamma$, and label points of interest (including the images of the multiples of $\pi/2$).

a) $f(z) = z$

b) $f(z) = 1/z$

c) $f(z) = 1/z^3$

d) $f(z) = z + 1/z$

e) $f(z) = 1 + z + z^2$

f) $f(z) = (z^2+1)/(z^2+2z-1)$

g) $f(z) = (z^2+2z-1)/(z^2+1)$

h) $f(z) = e^{Rz}$ for various real R in $[1, 100]$. What is happening near the origin? ◇

Exercise 2.3. Let $G = H \cup V$ be the following grid of segments in the unit square:

$$H = \{z \in \mathbb{C} \mid \textsc{Real}(z) \in [0,1], \textsc{Imag}(z) \in \{0, 1/3, 2/3, 1\}\}$$
$$V = \{z \in \mathbb{C} \mid \textsc{Imag}(z) \in [0,1], \textsc{Real}(z) \in \{0, 1/3, 2/3, 1\}\}$$

For each function f below, sketch the image $f(G)$. In your sketch, include the real and imaginary axes, draw $f(H)$ in blue, draw $f(V)$ in red, and label points of interest.

a) $f(z) = z^2$ b) $f(z) = z^3$ c) $f(z) = (iz)^3$

d) $f(z) = \textsc{Sqrt}(z)$ e) $f(z) = 1/z$ f) $f(z) = e^{2\pi i z}$ ◇

Exercise 2.4. Suppose that $f, g\colon \mathbb{C} \to \mathbb{C}$ are holomorphic. Prove, directly from the definitions, that $f+g$, $f \cdot g$, f/g, and $f \circ g$ are holomorphic. (For f/g we must avoid the zeros of g.) ◇

Exercise 2.5. Prove that $f(z) = \bar{z}$ is not holomorphic anywhere in $\mathbb{C}$. Using this, or otherwise, prove that $\textsc{Real}(z)$ and $\textsc{Imag}(z)$ are not holomorphic anywhere in $\mathbb{C}$. Similarly, prove that $|z|^2$ is holomorphic only at $z = 0$. ◇