Please post on the Q&A forum if any of the problems have mistakes, or are unclear, or can be improved. Thank you!

Exercise 3.1. For each (oriented) contour in Figure **??**, label the complementary regions with their winding numbers. ◇

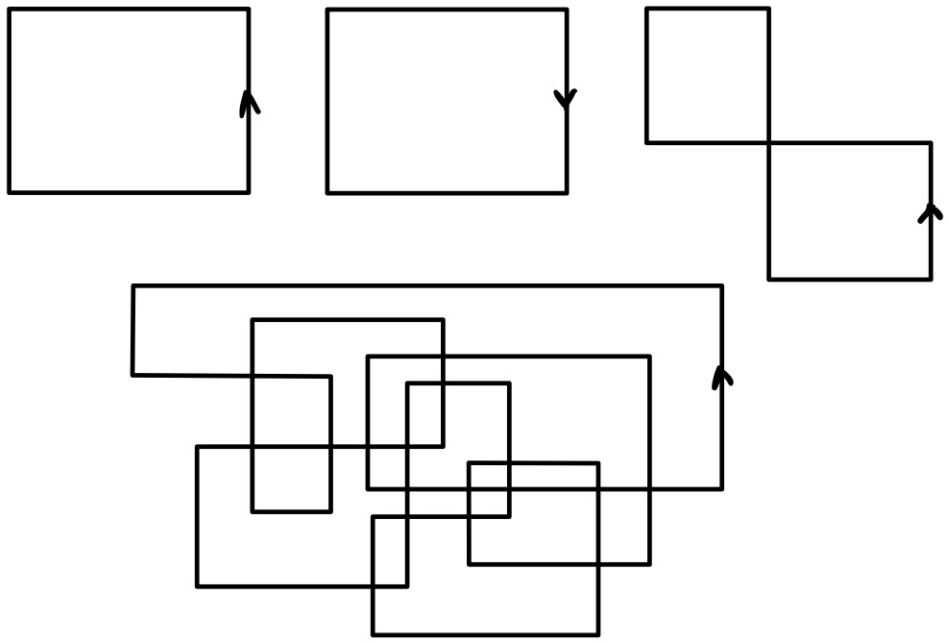


Figure 3.2: Four piecewise-linear contours.

Exercise 3.3. Suppose that $f(z) = 1/(1 - z + z^2)$. Find the series expansion of f about $z_0 = 0$ as well as its radius of convergence. ◇

Exercise 3.4. Suppose that $\textsc{Exp}, \textsc{Cos}, \textsc{Sin} \colon \mathbb{C} \to \mathbb{C}$ are the complex exponential and the complex trigonometric functions, each given by its power series. Prove the following directly from the definitions.

a) $\textsc{Exp}(0) = 1$.

b) $\textsc{Exp}(z + w) = \textsc{Exp}(z)\,\textsc{Exp}(w)$.

c) $\textsc{Exp}(iz) = \textsc{Cos}(z) + i\,\textsc{Sin}(z)$.

d) $\textsc{Exp}(z) = \textsc{Exp}(w)$ if and only if $z - w \in 2\pi i\mathbb{Z}$.

e) $\textsc{Exp}(\mathbb{C}) = \mathbb{C}^\times$. ◇

Exercise 3.5. Take $\omega = \textsc{Exp}(2\pi i/3)$; this is a primitive third root of unity. Let $T = T(1, \omega, \omega^2)$ be the triangle with vertices at 1, ω, and ω^2. Let $[1, \omega]$ be the line segment

from 1 to ω, parametrised by $t \mapsto 1 - t + t\omega$. Define and parametrise $[\omega, \omega^2]$ and $[\omega^2, 1]$ in similar fashion. For each contour γ, compute the contour integral $\int_\gamma \frac{dz}{z}$ directly from the definitions.

a) $\gamma = [1, \omega]$ b) $\gamma = [\omega, \omega^2]$ c) $\gamma = [\omega^2, 1]$ d) $\gamma = \partial T$ $\diamondsuit$

Exercise 3.6. Give a detailed proof that "holomorphic functions integrate to zero about rectangles". That is: suppose that $U \subset \mathbb{C}$ is a domain, suppose that $f \colon U \to \mathbb{C}$ is holomorphic, and suppose that $R \subset U$ is a rectangle. Show that $\int_{\partial R} f\, dz = 0$. (A proof was sketched in lecture (2026-10-20) and in Solution 6.8 of the LaTeXed lecture notes.) $\diamondsuit$