

MA3B8: COMPLEX ANALYSIS

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0. INTRODUCTION

...the shortest and best way between two truths of the real domain often passes through the imaginary one.

– Jacques Hadamard

The psychology of invention in the mathematical field

These notes set out the “material of record” for the 2026-2027 module MA3B8 (complex analysis) at Warwick. We give solutions to all exercises (except one) in Appendix A (starting on page 14). We also give a short index of notation (on page 21) and a list of references (on page 23).

If you find any errors in the notes, or any possible improvements, please inform me directly (at s.schleimer@warwick.ac.uk) or use the anonymous form:

<https://forms.gle/AnXkWf5DM8epZgF8A>

Acknowledgements. I thank the many teachers, colleagues, and students without whom these notes would not exist. I owe a significant debt to Donald Sarason and Curt McMullen; I first learnt this remarkable subject when taking their courses at UC Berkeley. Sarason’s book [12] gives a wonderful exposition of the topic for advanced undergraduates. I have also spent many happy hours reading Ahlfors’ classic text [1]. Those two books are the foundation underlying the house inside of which these notes were written.

I have also benefited directly from many enlightening conversations with Ed Brambley, Jeremy Kahn, Diane MacLagan, Cagri Sert, and Peter Topping.

AI declaration. I wrote the first version of the notes, for the 2025-26 academic year, without any AI. (I am particularly grateful to the students who pointed out mistakes last year!) This year I used AI as a line-by-line copyeditor and also as an occasional conversational partner: a replacement for my human colleagues, after they tired of hearing me go on (and on) about Cauchy’s theorem. All of the text was written, and all of the figures were drawn, by me.

1. RECTANGULAR AND POLAR COORDINATES

1.1. **Notation.** We use the following:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the natural numbers.
- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are, respectively, the integers, rationals, and reals.
- $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ is the complex numbers.

We assume the usual algebraic properties of each of these.

We assume the usual topological and geometric properties of the plane; this includes the idea of homeomorphism of subsets. When convenient we will identify $\mathbb{R}$ with either the x -axis in $\mathbb{R}^2$ or the real axis in $\mathbb{C}$.

We assume that the usual real-valued functions (cosine, sine, exponential, logarithm, square-root) with their usual properties are available. We also assume the usual analytic properties of real-valued functions of several variables, giving references where needed.

1.2. **Rectangular coordinates.** Suppose that

$$z = x + iy$$

is a complex number. Then we define

$$\text{REAL}(z) = x \quad \text{and} \quad \text{IMAG}(z) = y$$

to be the *real* and *imaginary* parts of z . We also define

$$\bar{z} = x - iy$$

to be the *complex conjugate* of z . We define

$$|z| = \sqrt{x^2 + y^2}$$

to be the *magnitude* of z . Note that $z\bar{z} = |z|^2$. Also, $|z| = 0$ if and only if $z = 0$.

Suppose that $r > 0$ is a positive real number. We define

$$B(z; r) = \{w \in \mathbb{C} : |w - z| < r\}$$

and

$$\overline{B(z; r)} = \{w \in \mathbb{C} : |w - z| \leq r\}$$

Thus $B(z; r)$ is the open, and $\overline{B(z; r)}$ is the closed, *ball* of radius r about z . (These are also called the open and closed *discs* of radius r about z .) We call $B^\times(z; r) = B(z; r) - \{z\}$ the *punctured* open ball of radius r about z . Similarly we call $\mathbb{C}^\times = \mathbb{C} - \{0\}$ the *punctured* plane.

Suppose that P is a non-empty subset of the plane $\mathbb{C}$. The *interior* of P , denoted P° , consists of those points $z \in P$ so that there is some $r > 0$ with $B(z; r) \subset P$. The *closure* of P , denoted $\bar{P}$, is the complement of the interior of $\mathbb{C} - P$. Said another way, the interior of P is the largest open subset of P while the closure of P is the smallest closed superset of P . The *topological boundary* of P is $\partial P = \bar{P} - P^\circ$: the closure of P minus the interior of P .

We define

$$C(z; r) = \{w \in \mathbb{C} : |w - z| = r\}$$

Thus $C(z; r)$ is the *circle* of radius r about z . Note that $C(z; r)$ is the topological boundary of both $B(z; r)$ and $\overline{B(z; r)}$. As a bit of notation we take $\mathbb{D} = B(0; 1)$ and $S^1 = C(0; 1)$; these are the (*open*) *unit disc* and the *unit circle*. We also use $\mathbb{D}^\times = \mathbb{D} - \{0\}$ for the *punctured unit disc*. We also define $\mathbb{H} = \{z \in \mathbb{C} \mid \text{IMAG}(z) > 0\}$: the *upper-half plane*.

Suppose that Z is a subset of the plane. We define

$$N(Z; r) = \{w \in \mathbb{C} : |w - z| < r \text{ for some } z \in Z\}$$

to be the *open r -neighbourhood* of Z . So $N(Z; r)$ is the union of the open r -balls about points z of Z .

1.3. Polar coordinates. Suppose that $z = x+iy \neq 0$. Thus $x^2+y^2 > 0$. So we may form

$$C = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad S = \frac{y}{\sqrt{x^2+y^2}}$$

Note that $C^2 + S^2 = 1$. We say that $\theta \in \mathbb{R}$ is an *argument* of z if

$$\cos(\theta) = C \quad \text{and} \quad \sin(\theta) = S$$

We define $\text{ARG}(z)$ to be the set of arguments of z .

Exercise 1.4. Suppose that $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Prove that $\text{ARG}(z)$ is non-empty. Prove that, for any argument θ , we have $\text{ARG}(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$. $\diamond$

Thus the argument is a *multi-valued function*: a one-to-many relation that is defined everywhere on its domain. This subtlety lies at the heart of the theory of Riemann surfaces.

Immediately from the definitions we deduce that, for any $z \in \mathbb{C}^\times$ and for any $\theta \in \text{ARG}(z)$, there is a *polar form*:

$$z = |z|(\cos(\theta) + i \sin(\theta))$$

As a useful bit of notation we write $e^{i\theta} = \cos(\theta) + i \sin(\theta)$. The addition law implies that $e^{i\theta+i\eta} = e^{i\theta}e^{i\eta}$.

Definition 1.5. Fix a positive integer n . Set $\omega_n = e^{2\pi i/n}$. Define

$$U_n = \{\omega_n^k \mid k \in \mathbb{Z}\}$$

This is the set of *n th roots of unity*. $\diamond$

By the addition law, we have $\omega_n^n = 1$ and that, under multiplication, U_n is a cyclic group of order n .

Exercise 1.6. Suppose that $z, w \in \mathbb{C}^\times$. Prove that

$$|zw| = |z||w| \quad \text{and} \quad \text{ARG}(zw) = \text{ARG}(z) + \text{ARG}(w) \quad \diamond$$

Exercise 1.7. Suppose that $z, w \in \mathbb{C}^\times$. Suppose that θ and η are arguments of z and w , respectively. Prove the cosine law

$$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta)$$

directly from the definitions and the addition law. $\diamond$

With arguments and the cosine law in hand, we can identify the geometry of $\mathbb{C}$ with that of $\mathbb{R}^2$.

Definition 1.8. Suppose that a and b are complex numbers, with $a \neq 0$. Suppose that $f: \mathbb{C} \rightarrow \mathbb{C}$ is given by $z \mapsto az + b$. Then we call f an (*orientation preserving*) *similarity*. If $|a| = 1$, then f is an (*orientation preserving*) *congruence*. If $a = 1$, then f is a *translation*. If $b = 0$, then f is a *homothety*¹ about the origin. If a is real and positive, and $b = 0$, then f is a *dilation* about the origin. If $|a| = 1$ and $b = 0$, then f is a *rotation* about the origin. $\diamond$

Exercise 1.9. Show that $\text{SIM}(\mathbb{C})$, the set of similarities of $\mathbb{C}$ together with the operation of function composition, is a group. Show that each subclass (congruence, translation, homothety, dilation, and rotation) gives a subgroup. $\diamond$

Suppose that U and V are subsets of the plane. We say that U and V are *similar* if there is a similarity sending U to V . We make the same definition for the various subclasses in Definition 1.8.

Exercise 1.10. Suppose that $z, w \in \mathbb{C}$. Suppose that $z \neq 0$ and that $|z - w| < |z|$. Prove that $w \neq 0$. Prove, furthermore, that for any $\theta \in \text{ARG}(z)$ there is a unique $\eta \in \text{ARG}(w)$ so that $|\theta - \eta| < \pi/2$. $\diamond$

We now prove that “the” argument is “continuous” on the punctured plane $\mathbb{C}^\times$.

Lemma 1.11. Suppose that $z \in \mathbb{C}^\times$. Suppose that $\epsilon \in (0, \pi/2)$. Then there is a $\delta > 0$ with the following property.

Suppose that $w \in \mathbb{C}$ has $|z - w| < \delta$. Then $w \neq 0$ and, for every $\theta \in \text{ARG}(z)$, there is a unique $\eta \in \text{ARG}(w)$ so that $|\theta - \eta| < \epsilon$.

Proof. We take $\delta = \sin(\epsilon)|z|$. Thus δ is positive and less than $|z|$. So $|z - w| < \delta < |z|$. By Exercise 1.10 we have $w \neq 0$ and we have a unique $\eta \in \text{ARG}(w)$ so that $|\theta - \eta| < \pi/2$.

Since $|z - w|^2 < \delta^2 = |z|^2 \sin^2(\epsilon)$ the cosine law gives us

$$|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta) < |z|^2 \sin^2(\epsilon)$$

Moving terms and simplifying gives

$$|z|^2 \cos^2(\epsilon) + |w|^2 < 2|z||w|\cos(\theta - \eta)$$

Subtracting from both sides gives

$$|z|^2 \cos^2(\epsilon) - 2|z||w|\cos(\epsilon) + |w|^2 < 2|z||w|\cos(\theta - \eta) - 2|z||w|\cos(\epsilon)$$

¹That is, scaling-and-rotation. A clearer (but longer) name would be *complex homothety*.

Thus

$$(|z| \cos(\epsilon) - |w|)^2 < 2|z||w|(\cos(\theta - \eta) - \cos(\epsilon))$$

The left-hand side is non-negative. Replacing it by zero and simplifying gives

$$\cos(\epsilon) < \cos(\theta - \eta)$$

Since cosine is even, and decreasing on $[0, \pi]$, we deduce that $|\theta - \eta| < \epsilon$, as desired. $\square$

2. CONTOUR INTEGRALS

2.1. Domains.

Definition 2.2. Suppose that $U \subset \mathbb{C}$ is non-empty, open, and connected. Then we call U a *domain*. $\diamond$

As suggested by Figure 2.3, domains can be quite wild.

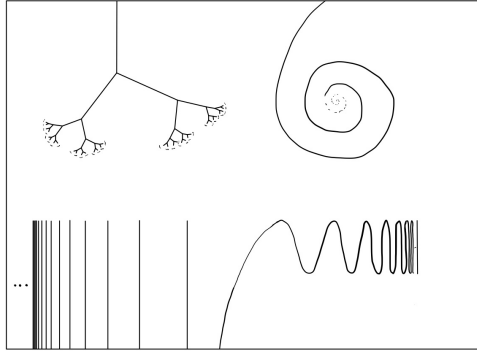


FIGURE 2.3. A wild domain. Here U is the open rectangle minus four closed sets: a *dendrite* (upper left), a *spiral* (upper right), a *comb* (lower left), and a *topologist's sine curve* (lower right).

Nonetheless, domains have several nice properties.

Exercise 2.4. Domains are path-connected. $\diamond$

(More is true; see Lemma ??.)

Exercise 2.5. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $P \subset U$ is compact and non-empty. Prove that there is a constant $\epsilon > 0$ so that the open neighbourhood $N(P; \epsilon)$ is contained in U . $\diamond$

2.6. Branches. With domains in hand, we give one solution to the “problem” of multi-valued functions.

Definition 2.7. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that F is a multi-valued function on U . A *branch* of F is

- a domain $V \subset U$ and
- a continuous function $f: V \rightarrow \mathbb{C}$

so that, for all $z \in V$, we have $f(z) \in F(z)$. $\diamond$

For example, ARG has a branch $\arg: \mathbb{H} \rightarrow \mathbb{R}$ obtained by always picking $\arg(z) \in (0, \pi)$. Here, the continuity of $\arg$ follows from Lemma 1.11.

Definition 2.8. We define a multi-valued function on $\mathbb{C}^\times$ as follows:

$$\text{SQRT}(z) = \{\sqrt{|z|}e^{i\theta/2}: \theta \in \text{ARG}(z)\}$$

We call the elements of $\text{SQRT}(z)$ the *square roots* of z . $\diamond$

Exercise 2.9. Prove that z has exactly two square roots; prove that their squares equal z .

Suppose that $B = B(z_0; r_0)$ is a ball with $0 < r_0 < |z_0|$. Prove that SQRT has exactly two branches with domain equal to B . $\diamond$

We return to this topic in Section ?? where we construct holomorphic branches of the logarithm.

2.10. Contours.

*I keep a close watch on this heart of mine
I keep my eyes wide open all the time
I keep the ends out for the tie that binds
Because you're mine, I walk the line*

– Johnny Cash
I walk the line

Definition 2.11. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $[a, b]$ is a closed non-trivial interval in $\mathbb{R}$.² A *contour in U* is a piecewise C^1 function $\gamma: [a, b] \rightarrow U$. That is, we have a finite sequence

$$(a = t_0 < t_1 < \dots < t_{n-1} < t_n = b) \subset [a, b]$$

so that, for all k , the restriction $\gamma_k = \gamma|_{[t_k, t_{k+1}]}$ is continuously differentiable. We call $\gamma_k = \gamma|_{[t_k, t_{k+1}]}$ a *piece* of γ . We call the point t_k a *breakpoint* and its image $\gamma(t_k)$ a *corner* of γ . $\diamond$

Note that we do not require the left and right derivatives to agree at breakpoints.

Note that some authors (for example [13, page 19]) additionally assume that all contours have non-vanishing derivative. We do not do that; instead we add the hypothesis once (to Definition ??) and use it once (in Lemma ??).

Example 2.12. Take $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$, a third root of unity. We consider the contour $\gamma: [0, 3] \rightarrow \mathbb{C}$ given by:

$$\gamma(t) = \begin{cases} (1-t) \cdot 1 + (t-0) \cdot \omega, & 0 \leq t \leq 1 \\ (2-t) \cdot \omega + (t-1) \cdot \omega^2, & 1 \leq t \leq 2 \\ (3-t) \cdot \omega^2 + (t-2) \cdot 1, & 2 \leq t \leq 3 \end{cases}$$

²It will be convenient to also allow *trivial contours*, where $a = b$. A trivial contour is a bit like the number zero: absurd, but it makes everything a bit more tidy.

This contour traces the boundary of an equilateral triangle with vertices at U_3 , the third roots of unity. So it has three pieces, its breakpoints are at the integers, and its corners are at the roots of unity. Note that the right and left derivative disagree at the breakpoints. $\diamond$

Suppose that $\gamma: [a, b] \rightarrow U$ is a contour. We say that γ is *closed* if $\gamma(b) = \gamma(a)$. (This is not the same as the topological notion of a closed set.) We say that γ is *simple* if $\gamma(r) = \gamma(s)$ implies that $r = s$ or $\{r, s\} = \{a, b\}$. (That is, simple contours are either injective, or are closed and fail to be injective only at their endpoints.) The triangular contour just above is both simple and closed.

Example 2.13. Suppose that $\gamma: [0, b] \rightarrow \mathbb{C}$ is given by $\gamma(\theta) = e^{i\theta}$.

- If $b = \pi$, then γ is simple but is not closed.
- If $b = 2\pi$, then γ is simple and is closed.
- If $b = 3\pi$, then γ is not simple and is not closed.
- If $b = 4\pi$, then γ is not simple but is closed.

In each case the contour has only one piece. $\diamond$

2.14. Integrals: definition and examples.

Definition 2.15. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma: [a, b] \rightarrow U$ is a contour. Suppose that $f: U \rightarrow \mathbb{C}$ is a continuous function. We define the *contour integral* of f along γ as follows.

$$\int_{\gamma} f dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

If γ has multiple pieces then we break the right-hand side into the corresponding finite sum of integrals. $\diamond$

Here is our first, and most important, example of a contour integral.

Example 2.16. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma: [0, 2\pi] \rightarrow U$ is given by $\gamma(\theta) = e^{i\theta}$. Suppose that $f: U \rightarrow \mathbb{C}$ is given by $f(z) = 1/z$. Then we have:

$$\int_{\gamma} f dz = \int_{\gamma} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = i \int_0^{2\pi} d\theta = 2\pi i \quad \diamond$$

In Example 2.16 the contour “winds” once (positively) about the origin. (We give a precise definition of “winds” in Definition 3.7.) And, as it happens, the integrand has a *pole* – a particular kind of singularity – at the origin. (We give a precise definition of “pole” in Section ??.) If we change the nature of the pole then the answer changes.

Exercise 2.17. Suppose that U and γ are as in Example 2.16. Suppose that k is an integer. Suppose that $f_k: U \rightarrow \mathbb{C}$ is given by $f_k(z) = z^k$. Compute $I_k = \int_{\gamma} f_k dz$. $\diamond$

Exercise 2.18. Suppose that U and γ are as in Example 2.16. Suppose that f is a *Laurent polynomial over $\mathbb{C}$* : a finite sum $f = \sum_{k=m}^n a_k z^k$ where $m < 0 < n$ and all of the a_k are complex. Prove the following.

$$\int_{\gamma} f dz = 2\pi i a_{-1} \quad \diamond$$

Moving the contour and integrand by the same amount does not affect the integral. More interestingly, in this particular case, shrinking the contour also does not affect the integral.

Example 2.19. Suppose that w is a point of $\mathbb{C}$. Suppose that $U = \mathbb{C} - \{w\}$ is a punctured plane. Suppose that $r > 0$ is real. Suppose that $\gamma: [0, 2\pi] \rightarrow U$ is given by $\gamma(\theta) = w + re^{i\theta}$. So $\gamma'(\theta) = ire^{i\theta}$. Suppose that $f: U \rightarrow \mathbb{C}$ is given by $f(z) = 1/(z - w)$. Then we have:

$$\int_{\gamma} f dz = \int_{\gamma} \frac{dz}{z - w} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

If $k \neq -1$, and if we replace f by $(z - w)^k$, then as in Exercise 2.17 the integral vanishes. $\diamond$

If we move the contour so that it does not wind about the origin, but leave the integrand unchanged, then the integral again vanishes. However, the proof is more delicate.

Exercise 2.20. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma: [0, 2\pi] \rightarrow U$ is given by $\gamma(\theta) = 2 + e^{i\theta}$. Suppose that $f: U \rightarrow \mathbb{C}$ is given by $f(z) = 1/z$. Give a direct proof that $\int_{\gamma} f dz = 0$. $\diamond$

All of the above are shadows of one of the main results of this module: Cauchy's theorem (Theorem ??).

2.21. Integrals: linearities, deformations, and orientations. The contour integral is linear with respect to the integrand, as follows.

Lemma 2.22. Suppose that U , γ , and f are as in Definition 2.15. Suppose that $g: U \rightarrow \mathbb{C}$ is a continuous function. Suppose that $p, q \in \mathbb{C}$ are complex numbers. Then we have:

$$\int_{\gamma} (p \cdot f + q \cdot g) dz = p \int_{\gamma} f dz + q \int_{\gamma} g dz \quad \square$$

As an application of linearity, we can understand what happens when we shrink a round contour.

Exercise 2.23. Suppose that $f: \mathbb{C} \rightarrow \mathbb{C}$ is continuous. Fix w in $\mathbb{C}$. Suppose that $r > 0$ is real. Define $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ by $\gamma(\theta) = w + re^{i\theta}$. Now define

$$I(w; r) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz$$

Prove, with w fixed, that $I(w; r)$ tends to $f(w)$ as r tends to zero. $\diamond$

The contour integral is also linear with respect to the domain of integration, as follows.

Lemma 2.24. *Suppose that U , γ , and f are as in Definition 2.15. Suppose that c is a point of $[a, b]$. Define $\gamma_a = \gamma|_{[a, c]}$ and $\gamma_b = \gamma|_{[c, b]}$. Then we have:*

$$\int_{\gamma} f dz = \int_{\gamma_a} f dz + \int_{\gamma_b} f dz \quad \square$$

The contour integral is almost independent of parametrisation, as follows.

Lemma 2.25. *Suppose that U , γ , and f are as in Definition 2.15. Suppose that $\delta: [c, d] \rightarrow [a, b]$ is piecewise C^1 . Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have:*

$$\int_{\gamma \circ \delta} f dz = \int_{\gamma} f dz$$

If, on the other hand, $\delta(c) = b$ and $\delta(d) = a$, then

$$\int_{\gamma \circ \delta} f dz = - \int_{\gamma} f dz$$

In the first case we say that γ and $\gamma \circ \delta$ have the *same orientation*. In the second case we say that they have *opposite orientations*.

Proof of Lemma 2.25. We compute as follows.

$$\begin{aligned} \int_{\gamma \circ \delta} f dz &= \int_c^d f(\gamma(\delta(u))) (\gamma \circ \delta)'(u) du && \text{Definition 2.15} \\ &= \int_c^d f(\gamma(\delta(u))) \gamma'(\delta(u)) \delta'(u) du && \text{chain rule} \\ &= \int_a^b f(\gamma(t)) \gamma'(t) dt && \text{substitution} \\ &= \int_{\gamma} f dz && \text{Definition 2.15} \end{aligned}$$

If δ reverses the endpoints then we instead obtain the following.

$$\int_{\gamma \circ \delta} f dz = \int_b^a f(\gamma(t)) \gamma'(t) dt = - \int_{\gamma} f dz \quad \square$$

2.26. Arc-length. Suppose that Q is a geometric quantity such as arc-length, angle, area, or similar. Then the *element of Q* is the differential we integrate to obtain Q . For example, the *element of arc-length* [3, page 28] in the xy -plane is:

$$ds = \sqrt{dx^2 + dy^2}$$

We use this as follows.

Definition 2.27. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma: [a, b] \rightarrow U$ is a contour. We define the *arc-length* of γ as follows.

$$L(\gamma) = \int_{\gamma} \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)| dt \quad \diamond$$

Example 2.28. The upper half of the unit circle is parametrised by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, \pi]$. Since this has unit speed, we have the following.

$$L(\gamma) = \int_{\gamma} \sqrt{dx^2 + dy^2} = \int_0^{\pi} |\gamma'(\theta)| d\theta = \int_0^{\pi} d\theta = \pi$$

The upper half of the circle is also parametrised (with the opposite orientation) by

$$\beta(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)$$

for $t \in [-1, 1]$. Computing, we have the following.

$$\begin{aligned} L(\beta) &= \int_{\beta} \sqrt{dx^2 + dy^2} = \int_{-1}^1 |\beta'(t)| dt \\ &= \int_{-1}^1 2 \sqrt{\frac{(1-t^2)^2}{(1+t^2)^4} + \frac{(2t)^2}{(1+t^2)^4}} dt \\ &= \int_{-1}^1 \frac{2}{(1+t^2)^2} \sqrt{(1-t^2)^2 + (2t)^2} dt \\ &= \int_{-1}^1 \frac{2}{1+t^2} dt \quad \text{so substitute } t = \tan(\theta) \\ &= \int_{-\pi/4}^{\pi/4} 2 d\theta = \pi \end{aligned} \quad \diamond$$

As the above example suggests, the arc-length integral is independent of parametrisation.

Lemma 2.29. Suppose that U and γ are as in Definition 2.27. Suppose that $\delta: [c, d] \rightarrow [a, b]$ is piecewise C^1 and bijective. Then we have $L(\gamma \circ \delta) = L(\gamma)$.

Exercise 2.30. Prove Lemma 2.29. $\diamond$

We now have an easy but extremely useful bound: the *ML-inequality*.

Lemma 2.31. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma: [a, b] \rightarrow U$ is a contour. Suppose that $f: U \rightarrow \mathbb{C}$ is continuous. We take

$$M = \max\{|f(\gamma(t))| : t \in [a, b]\}$$

Then we have:

$$\left| \int_{\gamma} f dz \right| \leq M \cdot L(\gamma)$$

Exercise 2.32. Prove Lemma 2.31. $\diamond$

3. VARIATION OF ARGUMENT

Customer: *I'd like to have an argument, please.*

Receptionist: *Certainly sir. Have you been here before?*

Customer: *No, this is my first time.*

Receptionist: *I see. Do you want to have the full argument,
or were you thinking of taking a course?*

– Monty Python
Argument Clinic

We have two ways to understand the variation of argument: analytical and topological.

3.1. The analytical viewpoint. Define the *element of argument* by:

$$d\theta = \frac{x dy - y dx}{x^2 + y^2}$$

This can be derived by differentiating $\tan(\theta) = y/x$.

Definition 3.2. Suppose that $\gamma = (\gamma_x, \gamma_y): [a, b] \rightarrow \mathbb{C}^\times$ is a contour. We define the *variation of argument* of γ , about zero, as follows.

$$V(\gamma) = \int_{\gamma} \frac{x dy - y dx}{x^2 + y^2} = \int_a^b \frac{\gamma_x(t)\gamma'_y(t) - \gamma_y(t)\gamma'_x(t)}{|\gamma(t)|^2} dt \quad \diamond$$

For example, the variation of argument about zero for the upper half-circle (oriented anti-clockwise) is π ; the variation for the whole circle is 2π .

Lemma 3.3. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C}^\times$ is a contour. Suppose that $\delta: [c, d] \rightarrow [a, b]$ is piecewise C^1 . Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have $V(\gamma \circ \delta) = V(\gamma)$. If, instead, $\delta(c) = b$ and $\delta(d) = a$ then $V(\gamma \circ \delta) = -V(\gamma)$. $\square$

The proof is similar to that of Lemma 2.25 and we omit it. The next proposition is another step in the direction of finding holomorphic branches of the logarithm (Section ??); this should be compared to Lemma 1.11.

Proposition 3.4. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C}^\times$ is a contour. Suppose that $\theta_a \in \text{ARG}(\gamma(a))$ is an argument. Then there exists a unique continuous $\theta: [a, b] \rightarrow \mathbb{R}$ so that

- $\theta(a) = \theta_a$ and
- $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ for all $t \in [a, b]$.

For this function θ , we have $V(\gamma) = \theta(b) - \theta(a)$.

Proof. Induction on the number of corners reduces the general case to the case where there are no corners in the interior of γ . That is, we

may assume that γ is C^1 . For s in $[a, b]$ we define $\gamma_s = \gamma|_{[a, s]}$. Next we define

$$\theta(s) = \theta_a + V(\gamma_s) = \theta_a + \int_{\gamma_s} \frac{x dy - y dx}{x^2 + y^2}$$

We deduce from the definition that $V(\gamma) = \theta(b) - \theta(a)$. We also deduce, from the fundamental theorem of calculus, that θ is differentiable (as a real function of s). Thus θ is continuous.

We define $r(s) = |\gamma(s)|$. We now claim that $\gamma(s) = r(s)e^{i\theta(s)}$ for all $s \in [a, b]$. To prove this, consider

$$h(s) = \frac{r(s)e^{i\theta(s)}}{\gamma(s)}$$

Differentiating h we find that $h'(s) = 0$ for all s .

Exercise 3.5. Check this. ◇

Also $h(a) = 1$; thus $h(s) = 1$ for all s . This implies that $\gamma(s) = r(s)e^{i\theta(s)}$ for all s .

Suppose that $\eta: [a, b] \rightarrow \mathbb{R}$ is another such function. That is, $\eta(a) = \theta_a$ and $\gamma(t) = r(t)e^{i\eta(t)}$ for all $t \in [a, b]$. We deduce that $\theta(t) - \eta(t)$ is continuous and also lies in $2\pi\mathbb{Z}$ for all t . Thus $\theta(t) - \eta(t)$ is constant. However $\theta(a) - \eta(a) = 0$, so $\eta = \theta$, as desired. □

Lemma 3.6. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C}^\times$ is a closed contour. Then we have $V(\gamma) \in 2\pi\mathbb{Z}$.

Proof. Recall that $\gamma(b) = \gamma(a)$. Pick any argument $\theta_a \in \text{ARG}(\gamma(a))$; apply Proposition 3.4 to build the function θ . We deduce that $e^{i\theta(b)} = e^{i\theta(a)}$. The addition law, and periodicity, tells us that $V(\gamma) = \theta(b) - \theta(a)$ lies in $2\pi\mathbb{Z}$, as desired. □

Definition 3.7. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C}^\times$ is a closed contour. We define

$$\text{WIND}(\gamma) = V(\gamma)/2\pi$$

to be the *winding number* of γ about zero. ◇

It follows from Lemma 3.6 that the winding number is an integer.

Remark 3.8. Suppose that $z_0 \in \mathbb{C}$. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C} - \{z_0\}$ is a contour missing z_0 . Then we define $V(\gamma, z_0)$, the variation of argument of γ about z_0 ; it is the variation of argument of $t \mapsto \gamma(t) - z_0$ about zero.

Suppose that γ is closed (and, again, misses z_0). Then we define $\text{WIND}(\gamma, z_0)$, the winding number of γ about z_0 ; it is the winding number of $t \mapsto \gamma(t) - z_0$ about zero. ◇

3.9. The topological viewpoint.

Second proof of Proposition 3.4. Recall that the image of γ avoids the origin. For every $t \in [a, b]$ we choose some r_t so that (the closure of) the ball $B(\gamma(t); r_t)$ misses the origin. Let I_t be the component of $\gamma^{-1}(B(\gamma(t); r_t))$ that contains t . So $\mathcal{U} = \{I_t\}$ is an open cover of $[a, b]$ by intervals. Since $[a, b]$ is compact, we may choose a finite subcover $\mathcal{V} \subset \mathcal{U}$. Let $T \subset [a, b]$ be the finite set of times t so that I_t lies in $\mathcal{V}$.

We now define a continuous function $\theta: [a, b] \rightarrow \mathbb{R}$ by induction. We first define $\theta(a) = \theta_a$. Suppose now that we have defined the function $\theta|_{[a, c]}$ on the interval $[a, c] \subset [a, b]$. Suppose additionally that $\theta|_{[a, c]}$ is continuous and satisfies $\theta(s) \in \text{ARG}(\gamma(s))$ for all $s \in [a, c]$.

If $c = b$ then take $\theta = \theta|_{[a, b]}$ and the definition is complete. Suppose instead that $c < b$. Now, c lies in (the interior of) I_t for some $t \in T$. Let c' be the supremum of I_t . Note that $\gamma([c, c'])$ lies in (the closure of) $B(\gamma(t); r_t)$. We apply Exercise 1.10 and the triangle inequality: for all s in $[c, c']$ there is a unique argument $\theta(s) \in \text{ARG}(\gamma(s))$ so that $|\theta(s) - \theta(c)| < \pi$. This extends $\theta|_{[a, c]}$ to $[a, c']$. By Lemma 1.11 the extension $\theta|_{[a, c']}$ is continuous. By construction $\theta(s)$ lies in $\text{ARG}(\gamma(s))$ for all $s \in [a, c']$. Finally, since T is finite this process terminates, giving the desired $\theta|_{[a, b]}$.

The argument for the uniqueness of θ (subject to the condition that $\theta(a) = \theta_a$) is identical to one given above. $\square$

3.10. Element of the logarithm. We end this section with three more examples.

The *element of radius* is:

$$dr = \frac{x dx + y dy}{\sqrt{x^2 + y^2}}$$

This can be derived by differentiating $r^2 = x^2 + y^2$. Suppose that $\gamma: [a, b] \rightarrow \mathbb{C}^\times$ is a contour in the punctured plane. Then integrating the element of radius over γ gives the difference $|\gamma(b)| - |\gamma(a)|$.

The *element of logarithmic radius* is:

$$\frac{dr}{r} = \frac{x dx + y dy}{x^2 + y^2}$$

This can be derived by differentiating $\log(r)$. Integrating this over γ , a contour in $\mathbb{C}^\times$, gives the logarithm of the ratio $|\gamma(b)|/|\gamma(a)|$.

Finally, the *element of the logarithm* is:

$$\frac{dr}{r} + i d\theta = \frac{dz}{z}$$

This differential has already appeared in the solution of Exercise 3.5.

APPENDIX A. SOLUTIONS TO EXERCISES

Solution 1.4. Recall that $\cos: \mathbb{R} \rightarrow \mathbb{R}$ is continuous; also $\cos(0) = 1$ and $\cos(\pi) = -1$. Since $C^2 + S^2 = 1$ we have that $C \in [-1, 1]$. The intermediate value theorem gives some $\theta \in [0, \pi]$ so that $\cos(\theta) = C$. Note that $\cos(-\theta) = C$ as well. We deduce that $\sin^2(\theta) = S^2$. Taking square roots (and replacing θ by $-\theta$ if needed) proves that θ lies in $\text{ARG}(z)$.

Since cosine and sine are 2π -periodic, we deduce that $\theta + 2\pi k$ lies in $\text{ARG}(z)$ for all $k \in \mathbb{Z}$. Suppose that η is an element of $\text{ARG}(z)$. Using the addition law we have the following.

$$\begin{aligned}\cos(\theta - \eta) &= \cos(\theta)\cos(-\eta) - \sin(\theta)\sin(-\eta) \\ &= \cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta) \\ &= C^2 + S^2 \\ &= 1\end{aligned}$$

Thus $\theta - \eta$ lies in $2\pi\mathbb{Z}$, as desired. $\square$

Solution 1.6. Suppose that $z = x + iy$. Suppose that $w = a + ib$. So $zw = xa - yb + i(xb + ay)$. We now compute as follows:

$$\begin{aligned}|zw|^2 &= (xa - yb)^2 + (xb + ya)^2 \\ &= x^2a^2 + y^2b^2 - 2xyab + x^2b^2 + y^2a^2 + 2xyab \\ &= x^2a^2 + x^2b^2 + y^2a^2 + y^2b^2 \\ &= (x^2 + y^2)(a^2 + b^2) \\ &= |z|^2|w|^2\end{aligned}$$

Taking the positive square root of both sides gives $|zw| = |z||w|$, as desired.

By the first part of Exercise 1.4 we may choose θ and η to be arguments of z and w respectively. By the second part of Exercise 1.4 it suffices to prove that $\theta + \eta$ is an argument of zw . We have the following polar forms:

$$z = |z|(\cos(\theta) + i\sin(\theta)) \quad w = |w|(\cos(\eta) + i\sin(\eta))$$

Taking their product, and applying the addition rule, gives:

$$zw = |z||w|(\cos(\theta + \eta) + i\sin(\theta + \eta))$$

By the above we have $|z||w| = |zw|$. Thus $\theta + \eta$ is an argument of zw , as desired. $\square$

Solution 1.7. Suppose that $z = x + iy$ and $w = r + is$. We compute as follows.

$$\begin{aligned}
& |z|^2 + |w|^2 - 2|z||w| \cos(\theta - \eta) \\
&= |z|^2 + |w|^2 - 2|z||w|(\cos(\theta) \cos(\eta) + \sin(\theta) \sin(\eta)) \\
&= |z|^2 + |w|^2 - 2|z||w| \left(\frac{x}{|z|} \frac{r}{|w|} + \frac{y}{|z|} \frac{s}{|w|} \right) \\
&= |z|^2 + |w|^2 - 2(xr + ys) \\
&= x^2 + y^2 + r^2 + s^2 - 2(xr + ys) \\
&= x^2 - 2xr + r^2 + y^2 - 2ys + s^2 \\
&= (x - r)^2 + (y - s)^2 \\
&= |z - w|^2
\end{aligned}$$

□

Solution 1.9. Suppose that a, b, c , and d are complex numbers. Suppose that f and g are similarities with $f(z) = az + b$ and $g(z) = cz + d$. Then $f \circ g(z) = a(cz + d) + b = acz + (ad + b)$. Since a and c are non-zero, so is ac . Thus $f \circ g$ is again a similarity.

We obtain an identity by taking $a = 1$ and $b = 0$. We make g an inverse for f by setting $c = 1/a$ and $d = -b/a$. The group operation is associative because function composition is associative.

Suppose that $|a| = |c| = 1$. Thus $|ac| = 1$ and $f \circ g$ is a congruence. Also $|1/a| = 1$, so the inverse of f is a congruence.

More generally, if the condition on a is closed under multiplication and inversion, and b is general or zero, then we obtain a subgroup. □

Solution 1.10. If $w = 0$ then $|z - w| < |z|$ becomes $|z| < |z|$, a contradiction. So $w \neq 0$. In particular $\text{ARG}(w)$ is well-defined.

Suppose that $\theta \in \text{ARG}(z)$ is the given argument of z . Fix any argument $\eta' \in \text{ARG}(w)$. Applying the cosine law we have

$$|z|^2 + |w|^2 - 2|z||w| \cos(\theta - \eta') < |z|^2$$

Simplifying we find

$$\frac{|w|}{2|z|} < \cos(\theta - \eta')$$

Thus $\cos(\theta - \eta')$ is strictly positive. So, for some $k \in \mathbb{Z}$, we have $|\theta - \eta' - 2\pi k| < \pi/2$. By Exercise 1.4 we have $\eta = \eta' + 2\pi k \in \text{ARG}(w)$. Thus $|\theta - \eta| < \pi/2$, as desired.

Suppose now that $\eta'' \in \text{ARG}(w)$ also satisfies $|\theta - \eta''| < \pi/2$. Thus

$$\begin{aligned}
|\eta - \eta''| &= |\eta - \theta + \theta - \eta''| \\
&\leq |\eta - \theta| + |\theta - \eta''| \\
&< \pi/2 + \pi/2 \\
&= \pi
\end{aligned}$$

But the difference $\eta - \eta''$ is a multiple of 2π (Exercise 1.4). Thus $\eta = \eta''$, as desired. $\square$

Solution 2.4. Suppose that p is a point of U . Let U_p be the path-component of U that contains p . We now prove the contrapositive: if U_p is not all of U then U is not connected.

Since U is open, there is some $\epsilon > 0$ so that $B(p; \epsilon)$ is contained in U . Since $B(p; \epsilon)$ is path-connected, it is a subset of U_p . Similarly, U_p contains an open neighbourhood of all of its points. So U_p is open.

Define $U' = U - U_p$. Fix any q in U' . Since U is open there is some $\delta > 0$ so that $B = B(q; \delta)$ is contained in U . Note that, since q is not in U_p , we have that B is disjoint from U_p . Thus B is a subset of U' . So U' is open. Thus U is not connected. $\square$

Solution 2.5. If $U = \mathbb{C}$ then any constant $\epsilon > 0$ works. Suppose instead that U is a proper subset of $\mathbb{C}$. Suppose that q lies in $\mathbb{C} - U$. Since distance is continuous and P is compact, there is some point $p \in P$ which is as close as possible to q . Thus writing $d(q, P) = |q - p|$ is well-defined. Suppose, for a contradiction, that (q_n) is a sequence in $\mathbb{C} - U$ so that $d(q_n, P)$ tends to zero. Let p_n be a point of P closest to q_n . Since P is compact, the sequence (q_n) is bounded, so has a convergent subsequence, which we pass to. Since $\mathbb{C} - U$ is closed, the limit q_∞ lies in $\mathbb{C} - U$. We pass to a further subsequence to ensure that (p_n) also converges say to p_∞ in P . But then $|q_\infty - p_\infty| = 0$. Thus $q_\infty = p_\infty$. We deduce that P and $\mathbb{C} - U$ intersect, a contradiction. $\square$

Solution 2.9. Suppose that z lies in $\mathbb{C}^\times$. Since $|z|$ is positive it has a positive square root $\sqrt{|z|}$. Take any $\theta, \eta \in \text{ARG}(z)$. By Exercise 1.4, we have a unique $k \in \mathbb{Z}$ so that $\theta - \eta = 2\pi k$. If k is even, then $e^{i\eta/2} = e^{i\theta/2}$. If k is odd, then $e^{i\eta/2} = -e^{i\theta/2}$. Thus z has exactly two square roots. Finally, we compute $(\sqrt{|z|}e^{i\theta/2})^2 = |z|e^{i\theta} = z$.

We define a function $f: B \rightarrow \mathbb{C}$ as follows. Fix any $\theta_0 \in \text{ARG}(z_0)$. Suppose that w lies in B . By Exercise 1.10 there is a unique $\eta \in \text{ARG}(w)$ so that $|\theta_0 - \eta| < \pi/2$. We define $f(w) = |w|^{1/2}e^{i\eta/2}$. Suppose that $z \in B$ is another point. By Exercise 1.10 we have some $\theta \in \text{ARG}(z)$ so that $|\theta_0 - \theta| < \pi/2$. By the triangle inequality we have $|\eta - \theta| < \pi$. So by Exercise 1.4 we have that θ is the unique argument of z that is within π of η . The continuity of f now follows from the continuity of the real square root, the real sine and cosine, and Lemma 1.11. Thus f is a branch of SQRT.

Suppose that g is another branch of SQRT. So g is continuous and never zero. So f/g is continuous and squares to give the function that is identically one on B . Thus for all $z \in B$ we have $f(z)/g(z) = \pm 1$. Since f/g is continuous it is constant. Thus $-f$ is the only other branch of SQRT. $\square$

Solution 2.17. We have already dealt with the case $k = -1$ in Example 2.16. So suppose that $k \neq -1$. We compute as follows.

$$\begin{aligned}
 \int_{\gamma} z^k dz &= \int_0^{2\pi} e^{ki\theta} \cdot ie^{i\theta} d\theta \\
 &= i \int_0^{2\pi} e^{(k+1)i\theta} d\theta \\
 &= \frac{i}{(k+1)i} \left[e^{(k+1)i\theta} \right]_0^{2\pi} \\
 &= \frac{1}{k+1} (1 - 1) \\
 &= 0
 \end{aligned}$$

Here we used the fundamental theorem of calculus for the real-valued cosine and sine (not for the complex exponential). $\square$

Solution 2.18. This follows from Lemma 2.22 and Exercise 2.17. $\square$

Solution 2.20. We reparametrise γ to instead use the domain $[-\pi, \pi]$. We now compute as follows:

$$\begin{aligned}
 \int_{\gamma} f dz &= \int_{\gamma} \frac{dz}{z} \\
 &= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{2 + e^{i\theta}} d\theta \\
 &= i \int_{-\pi}^{\pi} \frac{e^{i\theta}}{2 + e^{i\theta}} \frac{2 + e^{-i\theta}}{2 + e^{-i\theta}} d\theta \\
 &= i \int_{-\pi}^{\pi} \frac{2e^{i\theta} + 1}{5 + 4\cos(\theta)} d\theta \\
 &= i \int_{-\pi}^{\pi} \frac{2\cos(\theta) + 2i\sin(\theta) + 1}{5 + 4\cos(\theta)} d\theta
 \end{aligned}$$

We consider the real and imaginary parts separately.

The real part equals:

$$-2 \int_{-\pi}^{\pi} \frac{\sin(\theta)}{5 + 4\cos(\theta)} d\theta$$

This is a definite integral with odd integrand and symmetric domain of integration. Thus the integral vanishes.

The imaginary part (after dividing by i) equals:

$$\int_{-\pi}^{\pi} \frac{1 + 2\cos(\theta)}{5 + 4\cos(\theta)} d\theta$$

We substitute using

$$\cos(\theta) = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad d\theta = \frac{2}{1 + t^2} dt$$

The imaginary part becomes:

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{1 + 2 \cos(\theta)}{5 + 4 \cos(\theta)} d\theta \\ &= \int_{-\infty}^{\infty} \frac{(1 + t^2) + 2(1 - t^2)}{5(1 + t^2) + 4(1 - t^2)} \frac{2}{1 + t^2} dt \\ &= \int_{-\infty}^{\infty} \frac{3 - t^2}{9 + t^2} \frac{2}{1 + t^2} dt \end{aligned}$$

Since the integrand is even, it suffices to show that the integral over $(0, \infty)$ vanishes. Comparing with $\int_1^{\infty} dt/t^2$ we find that the (improper) integral converges. The integrand changes sign at $t = \sqrt{3}$. This suggests we use the substitution $t = 3/u$ (so $dt = -3du/u^2$):

$$\begin{aligned} & \int_0^{\infty} \frac{3 - t^2}{9 + t^2} \frac{2}{1 + t^2} dt \\ &= \int_{\infty}^0 \frac{3 - 9/u^2}{9 + 9/u^2} \frac{2}{1 + 9/u^2} \frac{-3}{u^2} du \\ &= \int_{\infty}^0 \frac{3/u^2 - 1}{1 + 1/u^2} \frac{2}{1 + 9/u^2} \frac{1}{u^2} du \\ &= \int_{\infty}^0 \frac{3 - u^2}{u^2 + 1} \frac{2}{u^2 + 9} du \end{aligned}$$

Thus the integral equals its negative, and so vanishes.

As a sanity check we numerically estimate the integral over $(0, \sqrt{3})$ and over $(\sqrt{3}, \infty)$.

```
sage: var("t"); f = 2*(3 - t^2) / ((9 + t^2)*(1 + t^2))
sage: numerical_integral(f, 0, sqrt(3))
(0.5235987755982988, 5.813114163347719e-15)
sage: numerical_integral(f, sqrt(3), Infinity)
(-0.5235987755982999, 2.302746825788907e-10)
```

In each answer, the first entry is an estimate of the integral; the second is an estimate of the error. (Our answer raises a further question: why is the numerical integral so very close to $\pi/6$?) $\square$

Solution 2.23. Fix any $\epsilon > 0$. Since f is continuous, there is some $\delta > 0$ so that $|z - w| \leq \delta$ implies that $|f(z) - f(w)| \leq \epsilon$. So suppose

that $0 < r < \delta$ and compute as follows:

$$\begin{aligned}
|I(w; r) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz - f(w) \right| && \text{definition of } I(w; r) \\
&= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - w} dz - \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{z - w} dz \right| && \text{Example 2.19} \\
&= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(w)}{z - w} dz \right| && \text{Lemma 2.22} \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(w + re^{i\theta}) - f(w)}{re^{i\theta}} rie^{i\theta} d\theta \right| && \text{Definition 2.15} \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} (f(w + re^{i\theta}) - f(w)) d\theta \right| && \text{simplify} \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |(f(w + re^{i\theta}) - f(w))| d\theta && \text{triangle inequality} \\
&\leq \epsilon && \square
\end{aligned}$$

Solution 2.30. We use the notation $\gamma = (\gamma_x, \gamma_y)$.

Suppose that δ preserves *orientation*: that is, $\delta(c) = a$ and $\delta(d) = b$. Then δ' is non-negative. Suppose that δ reverses orientation: that is, $\delta(c) = b$ and $\delta(d) = a$. Then δ' is non-positive.

Suppose first that δ preserves orientation. Then we compute as follows.

$$\begin{aligned}
L(\gamma \circ \delta) &= \int_{\gamma \circ \delta} \sqrt{dx^2 + dy^2} \\
&= \int_c^d \sqrt{(\gamma'_x(\delta(u)))^2 (\delta'(u))^2 + (\gamma'_y(\delta(u)))^2 (\delta'(u))^2} du \\
&= \int_c^d \sqrt{(\gamma'_x(\delta(u)))^2 + (\gamma'_y(\delta(u)))^2} \cdot |\delta'(u)| du \\
&= \int_c^d \sqrt{(\gamma'_x(\delta(u)))^2 + (\gamma'_y(\delta(u)))^2} \cdot \delta'(u) du \\
&= \int_a^b \sqrt{(\gamma'_x(t))^2 + (\gamma'_y(t))^2} dt \\
&= \int_{\gamma} \sqrt{dx^2 + dy^2} \\
&= L(\gamma)
\end{aligned}$$

If δ reverses orientation, then the two minus signs cancel, and we obtain the same arc-length. $\square$

Solution 2.32. We compute as follows.

$$\left| \int_{\gamma} f dz \right| \leq \int_{\gamma} |f| |dz| \leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq M \cdot L(\gamma) \quad \square$$

Solution 3.5. Suppose that $\gamma(s) = x(s) + iy(s)$. We suppress the variable s to obtain $\gamma = x + iy$ and $\gamma' = x' + iy'$. We may also write $r = \sqrt{x^2 + y^2}$. Thus $r'/r = (xx' + yy')/(x^2 + y^2)$. The fundamental theorem of calculus gives $\theta' = (xy' - yx')/(x^2 + y^2)$. We compute:

$$\frac{\gamma'}{\gamma} = \frac{x' + iy'}{x + iy} = \frac{x' + iy'}{x + iy} \frac{(x - iy)}{(x - iy)} = \frac{xx' + yy' + i(xy' - yx')}{x^2 + y^2} = \frac{r'}{r} + i\theta'$$

Recall that $h = re^{i\theta}/\gamma$. We differentiate:

$$h' = \frac{r'e^{i\theta} + re^{i\theta}i\theta'}{\gamma} - \frac{re^{i\theta}\gamma'}{\gamma^2} = \frac{re^{i\theta}}{\gamma} \left[\frac{r'}{r} + i\theta' - \frac{\gamma'}{\gamma} \right] = 0 \quad \square$$

INDEX OF SYMBOLS

$\square$, end of proof/solution, Lemma 1.11, 5
 $\diamond$, end of example/definition, Exercise 1.4, 3
 $\operatorname{REAL}(z)$, real part, Section 1.2, 2
 $\operatorname{IMAG}(z)$, imaginary part, Section 1.2, 2
 $\bar{z}$, complex conjugate, Section 1.2, 2
 $|z|$, magnitude, Section 1.2, 2
 $B(z; r)$, open ball, Section 1.2, 2
 $\overline{B(z; r)}$, closed ball, Section 1.2, 2
 $B^\times(z; r)$, punctured ball, Section 1.2, 2
 $\mathbb{C}^\times$, punctured plane, Section 1.2, 2
 P° , interior, Section 1.2, 2
 $\overline{P}$, closure, Section 1.2, 2
 ∂P , topological boundary, Section 1.2, 2
 $C(z; r)$, circle, Section 1.2, 3
 $\mathbb{D}$, open unit disc, Section 1.2, 3
 $\mathbb{D}^\times$, punctured unit disc, Section 1.2, 3
 S^1 , unit circle, Section 1.2, 3
 $\mathbb{H}$, upper-half plane, Section 1.2, 3
 $N(Z; r)$, open neighbourhood, Section 1.2, 3
 $\operatorname{ARG}(z)$, arguments, Section 1.3, 3
 U_n , roots of unity, Section 1.3, 3
 γ , contour, Definition 2.11, 6
 $\int_\gamma f dz$, contour integral, Definition 2.15, 7
 $\operatorname{WIND}(\gamma, z_0)$, winding number, Remark 3.8, 12

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