

LECTURE 1 SAVL SCHLEIMER

ADMIN: ALL ON MOODLE PAGE + CLASS PAGE.

SEE LAST YEAR'S VERSION FOR WHAT THIS YEAR'S WILL EVENTUALLY LOOK LIKE.

PLEASE USE Q+A FORUM FOR Q+A!

QUESTIONS?

OUTLINE: CONTOUR INTEGRALS, HOLOMORPHIC FCNS, HOMOTOLOGY, CAUCHY'S "THMS", POLES, RESIDUE THM, CONTOUR INTS, CONFORMAL MAPPINGS, RIEMANN MAPPING THM.

(1) NOTATION: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

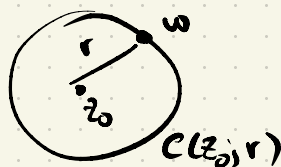
$$\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$$

$$z = x+iy, \operatorname{Re}(z) = x, \operatorname{Im}(z) = y, |z| = \sqrt{x^2+y^2}$$

$$\text{FOR } z_0 \in \mathbb{C}, r \in \mathbb{R}_{>0} \quad B(z_0; r) = \{w \in \mathbb{C} : |z_0 - w| < r\}$$

$$\overline{B(z_0; r)} = \{w \in \mathbb{C} : |z_0 - w| \leq r\}$$

$$C(z_0; r) = \{w \in \mathbb{C} : |z_0 - w| = r\}$$



(2) ARGUMENTS: SUPPOSE $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. PUNCTURED PLANE

$$\text{SUPPOSE } z = x+iy. \text{ DEFINE } C = x/\sqrt{x^2+y^2} \quad S = y/\sqrt{x^2+y^2}$$

SO: $C^2 + S^2 = 1$.

DEF:

$$\text{ARG}(z) = \{ \theta \in \mathbb{R} \mid \cos(\theta) = C \text{ AND } \sin(\theta) = S \}$$

IS THE SET of ARGUMENTS of z

EXERCISE: FOR $z, w \in \mathbb{C}^\times$

(i) $\text{ARG}(z) \neq \emptyset$

(ii) FOR $\theta \in \text{ARG}(z)$, $\text{ARG}(z) = \{ \theta + 2\pi k \mid k \in \mathbb{Z} \}$.

SO: $\text{ARG}: \mathbb{C}^\times \rightarrow \mathcal{P}(\mathbb{R})$ IS A "MULTI-VALUED FUNCTION".

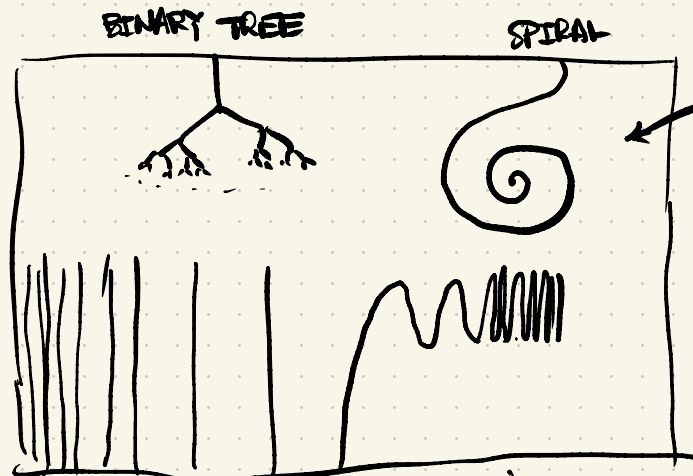
[KEY TRICKY POINT IN COMPLEX ANALYSIS]

(iii) $\text{ARG}(zw) = \text{ARG}(z) + \text{ARG}(w)$.

(3) DOMAINS: TO "SOLVE" THE "PROBLEM" of MULTI-VALUED FUNCTIONS

DEF: SUPPOSE $U \subset \mathbb{C}$ IS NONEMPTY, OPEN, CONNECTED.
THEN WE CALL U A DOMAIN.

CAUTION: DOMAINS CAN BE "WILD"



U IS RECTANGLE MINUS A "COMB", "SINE CURVE", "SPIRAL", AND "TREE".

COMB

TOPOLOGIST'S SINE CURVE

NONETHELESS: U IS BIHOLMORPHIC TO $D = B(0; 1)$
[UNIT DISC.]

LEMMA: SUPPOSE U IS A DOMAIN. THEN U IS
PATH-CONNECTED.

PROOF: PATH COMPONENTS ARE OPEN □

(4) BRANCHES

DEF: SUPPOSE $U, V \subset \mathbb{C}$ ARE DOMAINS, WITH $V \subset U$.

SUPPOSE $F: U \rightarrow \mathcal{P}(\mathbb{C})$ IS A MULTIVALUED FCN.

SAY $f: V \rightarrow \mathbb{C}$ IS A BRANCH of F (OVER V) IF

(i) $f(z) \in F(z)$ FOR ALL $z \in V$

(ii) f IS CONTINUOUS.

EXAMPLE: $\text{SQRT}: \mathbb{C}^* \rightarrow \mathcal{P}(\mathbb{C}^*)$ DEF BY

$$\begin{aligned}\text{SQRT}(z) &= \{w \in \mathbb{C} \mid w^2 = z\} \\ &= \{\sqrt{|z|}e^{i\theta/2} : \theta \in \text{ARG}(z)\}.\end{aligned}$$

HERE $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ [$\cos, \sin: \mathbb{R} \rightarrow \mathbb{R}$].

THERE IS NO BRANCH of SQRT OVER $\mathbb{C}^*$.

DEFINE $U = \mathbb{C}^*, V = \mathbb{C} - \mathbb{R}_{\leq 0}$

DEFINE $S: V \rightarrow \mathbb{C}$

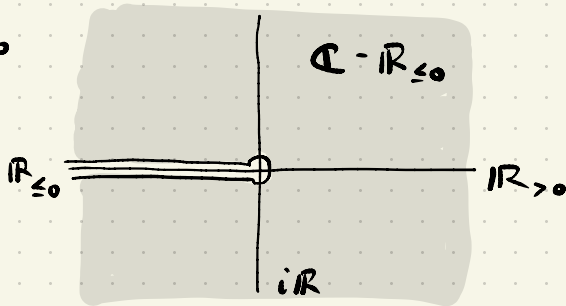
BY $S(z) = \sqrt{|z|} e^{i\theta/2}$

FOR $\theta \in \text{ARG}(z) \cap (-\pi, \pi)$

NOTE $-S(z)$ IS THE

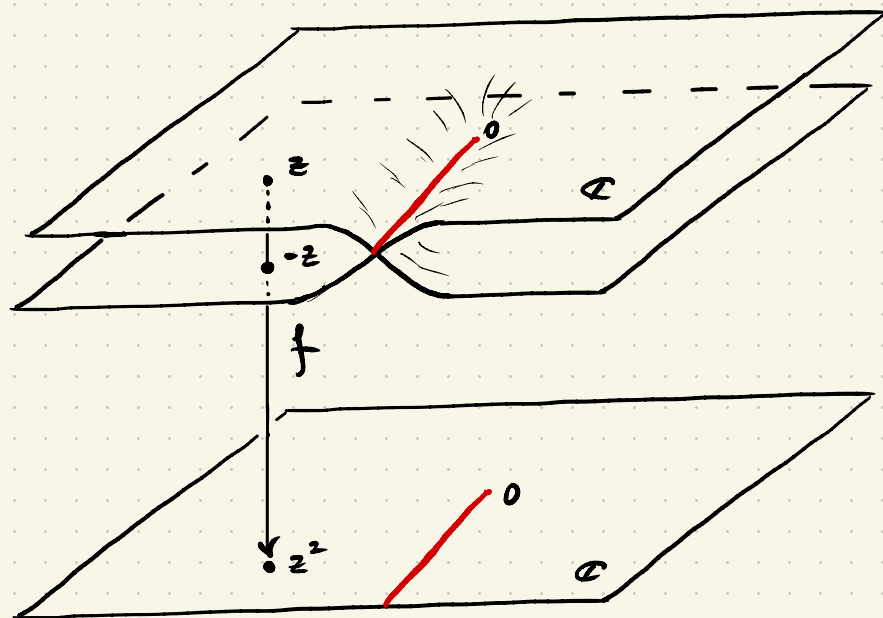
OTHER BRANCH of SQRT

[USES $\text{ARG}(z) \cap (\pi, 3\pi)$ INSTEAD!]



THE COLLECTION of ALL DOMAINS AND BRANCHES
PATCH TOGETHER TO FORM A RIEMANN SURFACE.

BONUS PICTURE of $f(z) = z^2$



NOTE: $f(z) = z^2$ IS TWO-TO-ONE FOR $z \in \mathbb{C}^*$.
SO THE UPPER RED RAY IS ACTUALLY TWO RAYS
OUT OF ZERO.