2026-10-07 MA388 LECTURE 2 SAUL SCHLEIMER

(1) CONTOURS

: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN.

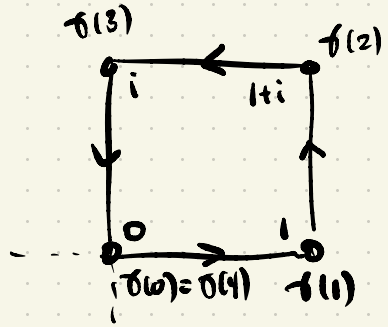
A CONTOUR IN U IS A PIECEWISE C^1 FUNCTION

$$\gamma : [a,b] \longrightarrow U.$$

THAT IS: THERE ARE $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ BREAKPOINTS SO THAT, FOR ALL k, THE PIECE $\gamma | [t_k, t_{k+1}]$ IS CONTINUOUSLY DIFFERENTIABLE. WE CALL THE POINTS $\gamma(t_k)$ CORNERS of γ.

EXAMPLE:

$\gamma : [0,4] \longrightarrow \mathbb{C}$ WITH

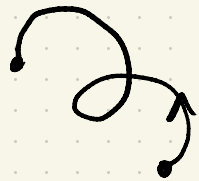


$$\gamma(t) = \begin{cases} t, & t \in [0,1] \\ 1 + i(t-1), & t \in [1,2] \\ 1 + i - (t-2), & t \in [2,3] \\ i(1-(t-3)), & t \in [3,4] \end{cases}$$

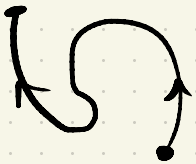
A CONTOUR IS CLOSED IF $\gamma(b) = \gamma(a)$.

" " " SIMPLE IF γ IS INJECTIVE

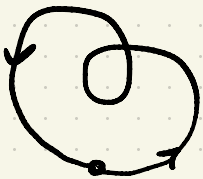
[EXCEPT PERHAPS $\gamma(a) = \gamma(b)$]



NOT CLOSED
NOT SIMPLE



NOT CLOSED
SIMPLE



CLOSED
NOT SIMPLE



CLOSED
SIMPLE.

② CONTOUR INTEGRALS

SUPPOSE $U \subset \mathbb{C}$ DOMAIN

SUPPOSE $f: U \to \mathbb{C}$ CONTINUOUS

SUPPOSE $\gamma: [a,b] \to \mathbb{C}$ CONTOUR.

DEFINE THE CONTOUR INTEGRAL (of f OVER γ)

$$\text{BY} \quad \int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

IF γ PIECEWISE THEN DEFINE VIA SUM OVER PIECES.

KEY EXAMPLES SET $U = \mathbb{C}^\times$, $\gamma: [0, 2\pi] \to \mathbb{C}$

$$\gamma(\theta) = e^{i\theta}$$

SET $f(z) = a + bz$ [FOR $a, b \in \mathbb{C}$]

SET $g(z) = 1/z$

SO:
$$\int_\gamma f\,dz = \int_0^{2\pi} (a + be^{i\theta})\, ie^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} a\,i\,e^{i\theta}\,d\theta + \int_0^{2\pi} b\,i\,e^{2i\theta}\,d\theta$$

$$= ai \int_0^{2\pi} (\cos(\theta) + i\sin(\theta))\,d\theta \to 0$$

$$+ bi \int_0^{2\pi} (\cos(2\theta) + i\sin(2\theta))\,d\theta \to 0$$

$$= 0.$$

$$\int_\gamma g\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{i\theta}}\, ie^{i\theta}\,d\theta = 2\pi i$$

CRUCIAL: γ CLOSED AND f HOLOMORPHIC

[AND g IS NOT: IT HAS A POLE AT $z=0$.]

DEFINITION: SUPPOSE $m, n \in \mathbb{Z}$ WITH $m < 0 < n$. SUPPOSE $a_k \in \mathbb{C}$ FOR $k \in [m,n]$. SET

$$f(z) = \sum_{k=m}^{n} a_k z^k.$$ CALL f A LAURENT POLYNOMIAL.

EX: $f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2$. SAY.

EXERCISE: SET $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma(\theta) = e^{i\theta}$

PROVE $\int_\gamma f\, dz = 2\pi i a_{-1}$.

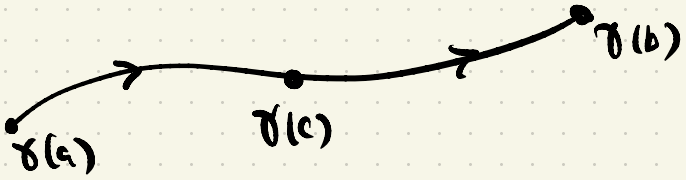
USE

LEMMA (LINEARITY I)

$$\int_\gamma (af + bg)\, dz = a \int_\gamma f\, dz + b \int_\gamma g\, dz.$$

LEMMA (LINEARITY II): SUPPOSE $\gamma : [a,b] \longrightarrow U$ SUPPOSE $c \in [a,b]$. THEN

$$\int_\gamma f\, dz = \int_{\gamma|[a,c]} f\, dz + \int_{\gamma[c,b]} f\, dz$$



(3) ARC-LENGTH : SUPPOSE $\gamma : [a,b] \to \mathbb{C}$ IS A CONTOUR.
WE WRITE $\gamma(t) = u(t) + i\,v(t)$, FOR $u, v : [a,b] \to \mathbb{R}$.

DEF: THE ELEMENT of ARC LENGTH IS

$$ds = \sqrt{dx^2 + dy^2}$$

IT GIVES THE ARC-LENGTH FUNCTIONAL

$$L(\gamma) = \int_\gamma ds = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\, dt$$

$$= \int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\; dt.$$