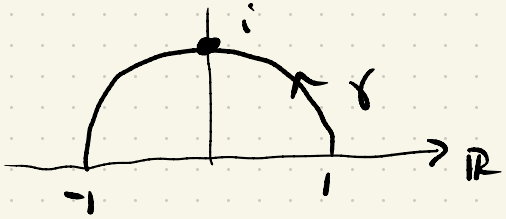
2026-10-09 MA388 LECTURE 3 SAUL SCHLEIMER

(1) ARC-LENGTH

$$L(\gamma) = \int_\gamma ds = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\, dt.$$

EXAMPLE: $\gamma(\theta) = e^{i\theta}$, $\gamma : [0, \pi] \longrightarrow \mathbb{C}$, $L(\gamma) = \pi$.



LEMMA: SUPPOSE $\gamma : [a,b] \to \mathbb{C}$ A CONTOUR. SUPPOSE $\delta : [c,d] \longrightarrow [a,b]$ PIECEWISE C^1 AND BIJECTIVE. THEN $L(\gamma \circ \delta) = L(\gamma)$.

SO REPARAMETRISATION IS "FREE"

(2) ML-INEQUALITY IS SUPER USEFUL

LEMMA: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $f : U \to \mathbb{C}$ IS CONTINUOUS. SUPPOSE $\gamma : [a,b] \to U$ A CONTOUR. DEFINE

$$M(f, \gamma) = \max \left\{ |f(\gamma(t))| : t \in [a,b] \right\}.$$

THEN:

$$\left| \int_\gamma f\, dz \right| \leq M \cdot L \quad \begin{cases} M = M(f, \gamma) \\ L = L(\gamma) \end{cases}$$

(3) ELEMENT of ARGUMENT

HERE "ELEMENT of $\mathbb{C}^\times$" MEANS:

(i) "A BIT of $\mathbb{C}^\times$"

(ii) THE THING YOU INTEGRATE TO GET $\mathbb{C}^\times$.

SO DEFINE

$$d\theta = \frac{x\,dy - y\,dx}{x^2+y^2}$$

($x\,dy - y\,dx$ ← DET $\begin{pmatrix} x & dx \\ y & dy \end{pmatrix}$; x^2+y^2 ← SCALE INVAR)

CAN DERIVE THIS BY DIFF'ING $\frac{y}{x} = \tan(\theta)$.

DEFINE, FOR CONTOUR $\gamma: [a,b] \longrightarrow \mathbb{C}^\times$,

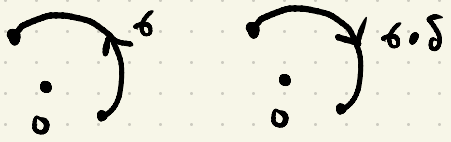
$$V(\gamma) = \int_\gamma d\theta = \int_\gamma \frac{x\,dy - y\,dx}{x^2+y^2}$$

THE VARIATION of ARGUMENT:

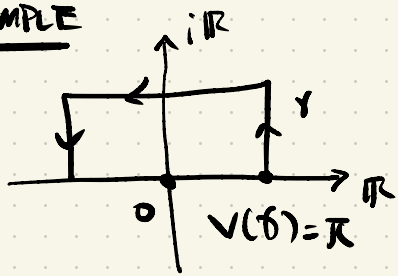
LEMMA: u, γ GIVEN. $\delta: [c,d] \longrightarrow [a,b]$ REPARAM.

THEN $$V(\gamma\circ\delta) = \begin{cases} V(\gamma), & \text{IF } \delta(c) = a \\ -V(\gamma), & \text{IF } \delta(c) = b. \end{cases}$$

PICTURE:



EXAMPLE



[ANGLE-LIFTING]

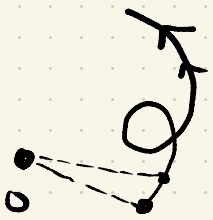
PROP: SUPPOSE $\gamma: [a,b] \longrightarrow \mathbb{C}^\times$ CONTOUR. FIX ANY $\theta_a \in \text{ARG}(\gamma(a))$. THEN THERE EXISTS A UNIQUE BRANCH $\theta: [a,b] \longrightarrow \mathbb{R}$ of ARG $\circ\,\gamma$ WITH $\theta(a) = \theta_a$

PROOF: WE NEED $\theta : [a,b] \longrightarrow \mathbb{R}$ CONTINUOUS

WITH (i) $\theta(a) = \theta_a$

(ii) $\theta(t) \in \text{ARG}(\gamma(t))$.

PICTURE



DEFINE $\theta(t) = \theta_a + V(\gamma|[a,t])$

SO CONTINUOUS B/C INTEGRAL

AND $\theta(a) = \theta_a$ BY CONSTRUCTION.

DEFINE $r(t) = |\gamma(t)|$.

DEFINE $h(t) = r(t) \cdot e^{i\theta(t)} / \gamma(t)$.

EXERCISE: $h'(t) = 0$.

ALSO $h(a) = 1$. SO $h(t) = 1$ FOR ALL t. $\square$.

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(4) ELEMENT of RADIUS

DEFINE $r = \sqrt{x^2+y^2}$ SO $r^2 = x^2 + y^2$

SO $2r\,dr = 2x\,dx + 2y\,dy$

SO $dr = \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ ← $(x,y)\cdot(dx,dy)$

← NOT SCALE INVAR.

EXERCISE

$$\int_\gamma dr = \int_\gamma \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}} = |\gamma(b)| - |\gamma(a)|$$

CHANGE IN RADIUS FOR $\gamma : [a,b] \longrightarrow \mathbb{C}^\times$