2026-10-13 LECTURE 4 MA3BB SAUL SCHLEIMER

(1) ELEMENT of LOG(z)

WE GIVE TWO DEFINITIONS

$$\frac{dz}{z} = \frac{dr}{r} + i\,d\theta$$

LEMMA: THESE AGREE

PROOF: WE COMPUTE:

$$\frac{dr}{r} + i\,d\theta = \frac{x\,dx + y\,dy}{x^2+y^2} + i\,\frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \frac{x\,dx + y\,dy + ix\,dy - iy\,dx}{(x+iy)(x-iy)}$$

$$= \frac{x\,(dx+i\,dy) - i^2 y\,(dy - i\,dx)}{(x+iy)(x-iy)}$$

$$= \frac{x\,(dx+i\,dy) - iy\,(i\,dy + dx)}{(x+iy)(x-iy)}$$

$$= \frac{dx + i\,dy}{x+iy} = \frac{dz}{z} \qquad \square$$

WE WILL RETURN, SEVERAL TIMES, TO THE INTEGRAL $\int \frac{dz}{z}$.

② HOLOMORPHICITY

DEF: SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ IS A FUNCTION. FIX $z_0 \in U$. WE SAY f IS **HOLOMORPHIC** AT z_0 IF

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{EXISTS.}$$

WE CALL THE LIMIT $f'(z_0)$. SAY f **HOLOMORPHIC** IF IT IS HOLOMORPHIC FOR ALL $z_0 \in U$. HERE WE OBTAIN $f': U \to \mathbb{C}$ THE **COMPLEX DERIVATIVE** OF f.

LEMMA: HOLOMORPHIC $\Rightarrow$ CONTINUOUS.

LEMMA: SUPPOSE $f, g: U \to \mathbb{C}$ HOLOMORPHIC. THEN SO IS $f+g$, $f \cdot g$, f/g [AWAY FROM ZEROS of g.]

LEMMA: SUPPOSE $U, V, W \subset \mathbb{C}$ DOMAINS. SUPPOSE $f: U \to V$, $g: V \to W$ HOLOMORPHIC. THEN SO IS $g \circ f: U \to W$.

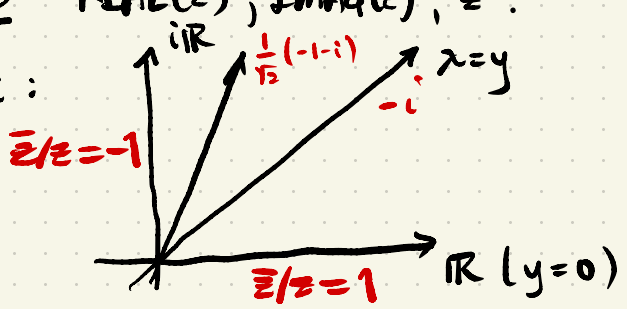
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③ EXAMPLES:

CONSTANT FUNS, IDENTITY FCNS POLYNOMIALS IN z, $f(z) = P(z)/Q(z)$ (RATIONAL) FOR P, Q POLYNOMIALS.

NONEXAMPLES $\mathrm{REAL}(z)$, $\mathrm{IMAG}(z)$, $\overline{z}$.

PROOF FOR $\overline{z}$:



$\overline{z}/z$ CONSTANT ON LINES

EXERCISE: DO THIS FOR $z \longmapsto \mathrm{REAL}(z)$.

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(4) LEMMA 4.8 SUPPOSE U A DOMAIN, $f: U \to \mathbb{C}$ HOLOMORPHIC. FIX $z_0 \in U$. DEFINE $\rho: U \to \mathbb{C}$ BY

$$\rho(z) = \begin{cases} 0, & z = z_0 \\ \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0), & z \neq z_0 \end{cases}$$

THEN: ρ IS CONTINUOUS.

PROOF: CLEAR FOR $z \neq z_0$, BY HOLOMOR. of f AT $z = z_0$ □

COROLLARY: U, f AS BEFORE. THEN THE TOTAL DERIVATIVE of f (AS FCN $\mathbb{R}^2 \to \mathbb{R}^2$) IS

$$Df(z_0) = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$$

FOR $r = |f'(z_0)|$

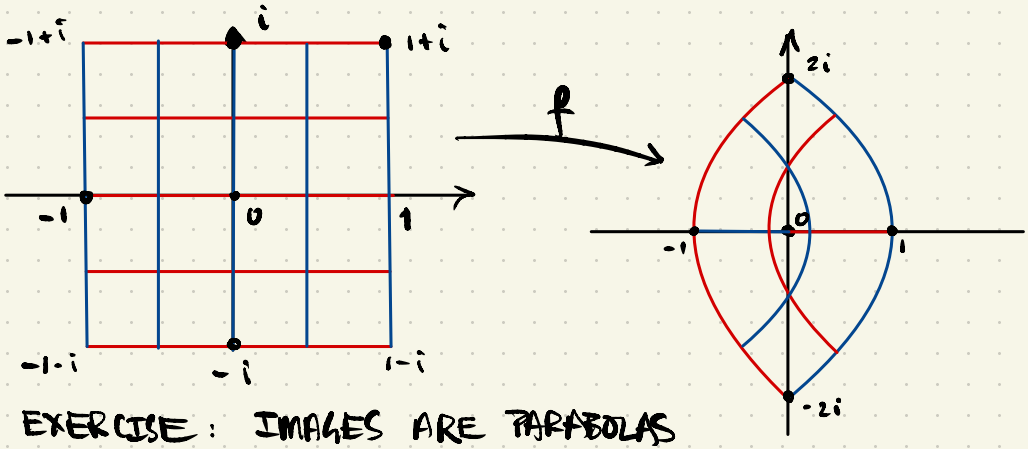
$\theta \in \mathrm{ARG}(f'(z_0))$

[IF $f'(z_0) \neq 0$].

DEDUCE: WHEN $f'(z_0) \neq 0$, f PRESERVES SIGNED

ANGLES. ALSO SAY f IS CONFORMAL.

⑤ SKETCHING CONSIDER $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$.



EXERCISE: IMAGES ARE PARABOLAS MEETING AT RIGHT ANGLES.

NOTE: f NOT CONFORMAL AT $z_0 = 0$ [B/C $f'(0) = 0$]

INSTEAD IT IS ANGLE DOUBLING

EXERCISE DRAW PREIMAGE of GRID UNDER $f(z) = z^2$.

[AKA IMAGE UNDER SQRT]

EXERCISE: $g(z) = z^3$. DRAW IMAGE of GRID.

