2026-10-14 LECTURE 5 MA3BB SAUL SCHLEIMER

(1) POWER SERIES: SUPPOSE $(a_k)_{k \in \mathbb{N}}$ IS A SEQUENCE, $a_k \in \mathbb{C}$. FIX $z_0 \in \mathbb{C}$. DEFINE

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

A (FORMAL) POWER SERIES

DEFINE $$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k}$$ [$R = 0, \infty$ ALLOWED]

WE CALL R THE RADIUS OF CONVERGENCE OF f.

DEFINE $f_N(z) = \sum_{k=0}^{N} a_k (z - z_0)^k$ THE N^{TH} PARTIAL SUM

SO GET USUAL PROPERTIES AS REAL CASE: [SEE NOTES!]

EXAMPLE: $f(z) = \sum_{k} (-1)^k z^{2k} = 1 - z^2 + z^4 - z^6 + \cdots$

HAS $R = 1$. NOTE $f(z) = \frac{1}{1+z^2}$ IN $B(0;1)$.

NOTE THAT $\frac{1}{1+z^2}$ BLOWS UP AT $z = i$ SO THE RADIUS OF CONVERGENCE COULDN'T BE ANY BIGGER.

EXERCISE: GIVE POWER SERIES EXPANSION (AT $z_0 = 0$) FOR $f(z) = \frac{1}{1+z+z^2}$. [HINTS ARE AVAILABLE!]

EXERCISE: GIVE POWER SERIES EXPANSION AT $z = 1$ FOR $f(z) = \frac{1}{1+z^2}$

(2) EXPONENTIAL AND TRIG.

WE DEFINE

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

AGREE WITH exp, cos, sin ON $\mathbb{R} \subset \mathbb{C}$.

EXERCISES: ① $\exp(z)$ HOLO, $\exp'(z) = \exp(z)$

② $\exp(z+w) = \exp(z)\exp(w)$

③ $\exp(iz) = \cos(z) + i\sin(z)$

④ $\exp(z) = 1$ IFF $z \in 2\pi i \mathbb{Z}$

⑤ $\exp(\mathbb{C}) = \mathbb{C}^\times$.

EXAMPLE: FIX $w \in \mathbb{C}$. DEFINE $F(z) = \exp(z)\exp(w-z)$

SO: $F(0) = 1 \cdot \exp(w) = \exp(w)$.

$$F'(z) = \exp(z)\cdot\exp(w-z) - \exp(z)\exp(w-z)$$
$$= 0.$$

SO F IS CONSTANT. SO $\exp(w) = \exp(z)\exp(w-z)$

NOW SET $w = z+u$

$$\boxed{\exp(z+u) = \exp(z)\exp(u)}$$

③ LOGARITHM $\log : \mathbb{C}^\times \longrightarrow \mathcal{P}(\mathbb{C})$

DEFINE: $\log(w) = \{ z \in \mathbb{C} \mid \exp(z) = w \}$

LEMMA: $\log(w) = \log(|w|) + i\,\mathrm{ARG}(w)$

PROOF: $\exp(\log|w| + i\theta)$ FOR $\theta \in \arg(w)$

$= \exp(\log|w|) \cdot \exp(i\theta)$

$= |w| \cdot (\cos\theta + i\sin\theta)$

$= |w| \cdot (\cos\theta + i\sin\theta)$

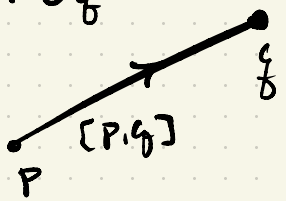
$= |w| e^{i\theta} = w$ □

SO: LOG IS "CONTINUOUS". WE WILL SHOW (MUCH LATER) THAT IT IS HOLOMORPHIC

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(4) SEGMENTS: SUPPOSE $p, q \in \mathbb{C}$. DEFINE $[p,q]$ TO BE THE LINE SEGMENT BETWEEN THEM. WE USE THE SAME NOTATION FOR THE CONTOUR

$\gamma : [0,1] \longrightarrow \mathbb{C}$; $\gamma(t) = (1-t)p + t \cdot q$

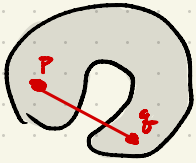


(5) CONVEXITY: SAY $K \subset \mathbb{C}$ IS CONVEX IF, FOR ALL $p, q \in K$ WE HAVE $[p,q] \subset K$.

EXAMPLES



NON-EXAMPLE



EXERCISE: $B(z_0, r)$ AND $\overline{B(z_0, r)}$ ARE BOTH CONVEX.

DEF: FOR $S \subset \mathbb{C}$, DEFINE

$$\text{HULL}(S) = \bigcap \left\{ K \subset \mathbb{C} \;\middle|\; \begin{matrix} S \subset K \\ K \text{ CONVEX} \end{matrix} \right\}$$

BONUS PICTURE of EXP AND BRANCH of LOG.

