2026-10-16 LECTURE 6 MA3B8 SAUL SCHLEIMER

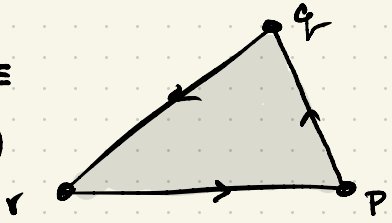
EXERCISE: $\mathbb{C}$ IS THE HULL of A COUNTABLE SET.
$\overline{B(0;1)}$ IS NOT!

① TRIANGLES

DEF: SUPPOSE $p, q, r \in \mathbb{C}$. DEFINE

$$T(p,q,r) = \text{HULL}(\{p,q,r\})$$

SET $T = T(p,q,r)$.



SO $\partial T = [p,q] \cup [q,r] \cup [r,p]$ [AS SETS]

THIS IS THE TOPOLOGICAL BOUNDARY. WE DEFINE THE POLYGONAL BOUNDARY ∂T TO BE THE CONCATENATION of THE CONTOURS $[p,q], [q,r], [r,p]$ IN THAT ORDER.

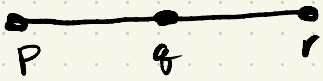
THREE KINDS of TRIANGLE

POSITIVE



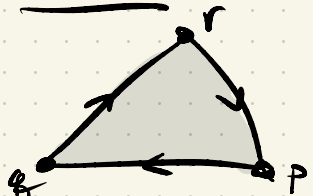
ANTI-CLOCKWISE

DEGENERATE



COLINEAR

NEGATIVE



CLOCKWISE

② INTEGRALS ABOUT TRIANGLES.

EXERCISES: SUPPOSE $T = T(p,q,r)$ IS A TRIANGLE

(1) $\int_{\partial T} dz = 0$ (2) $\int_{\partial T} z\,dz = 0$

(3) $\int_{\partial T} f\,dz = 0$ FOR $f \in \mathbb{C}[z]$ POLYNOMIAL

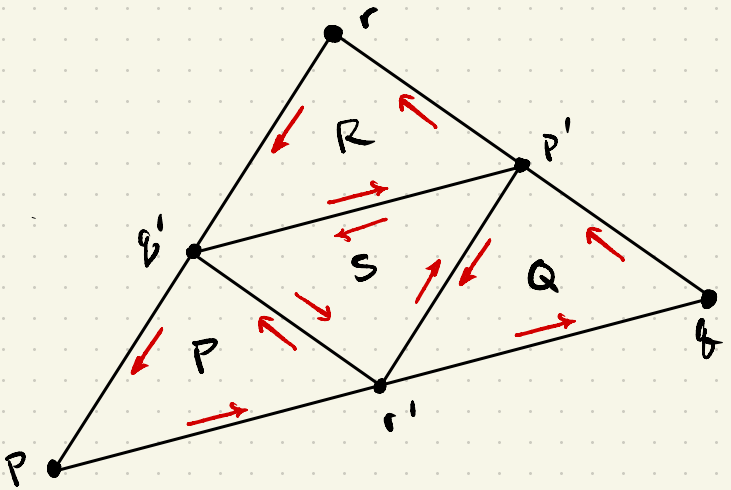
(4) $\int_{\partial T} f\,dz = 0$ FOR f ANALYTIC PROOF $f_N \to f$ UNIFORMLY

③ MIDPOINT SUBDIVISION

SUPPOSE $T = T(p,q,r)$. DEFINE MIDPOINTS

$$p' = \frac{q+r}{2} \qquad q' = \frac{r+p}{2} \qquad r' = \frac{p+q}{2}$$

AND TRIANGLES P, Q, R, S AS SHOWN:



DEFINE

$MID(T) = \{P,Q,R,S\}$

THE MIDPOINT SUBDIVISION of $T = T(p,q,r)$

RECALL $L(\partial T)$ = ARC-LENGTH of ∂T

DEFINE $D(\partial T)$ = DIAMETER of ∂T

$$= \max\{|z-w| \mid z,w \in \partial T\}$$

NOTE: $L(\partial P) = L(\partial Q) = L(\partial R) = L(\partial S) = \frac{1}{2} L(\partial T)$

$D(\partial P) = D(\partial Q) = D(\partial R) = D(\partial S) = \frac{1}{2} D(\partial T)$

---

④ GOURSAT'S LEMMA: SUPPOSE U A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $T = T(p,q,r) \subset U$ A TRIANGLE. THEN $\int_{\partial T} f\,dz = 0$.

"HOLOMORPHIC FCNS INTEGRATE TO ZERO ABOUT TRIANGLES"

PROOF: FUNCTION f NOW FIXED. SO DEFINE, FOR CONTOUR $\gamma : [a,b] \to \mathbb{C}$ INTEGRAL $I(\gamma) = \int_\gamma f\, dz$.

NOTE: $I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$
DUE TO THE CANCELLING of INTERNAL EDGES
[OPPOSITE ORIENTATIONS GIVE OPPOSITE SIGNS]
THUS $\frac{1}{4}|I(\partial T)| \le \text{MAX}\{|I(\partial P)|, \cdots, |I(\partial S)|\}$

NOW RECURSE: SET $T_0 = T$.
PICK T_{n+1} ANY TRIANGLE IN $\text{MID}(T_n)$ SO THAT

$$|I(\partial T_{n+1})| = \text{MAX}\{|I(\partial P)| : P \in \text{MID}(T_n)\}$$

SO HAVE $\frac{1}{4}|I(\partial T_n)| \le |I(\partial T_{n+1})|$

$$\tfrac{1}{2} L(\partial T_n) = L(\partial T_{n+1})$$

$$\tfrac{1}{2} D(\partial T_n) = D(\partial T_{n+1})$$

THUS: $\frac{1}{4^n}|I(\partial T_0)| \le |I(\partial T_n)|$ (*)

$$\tfrac{1}{2^n} L(\partial T_0) = L(\partial T_n)$$

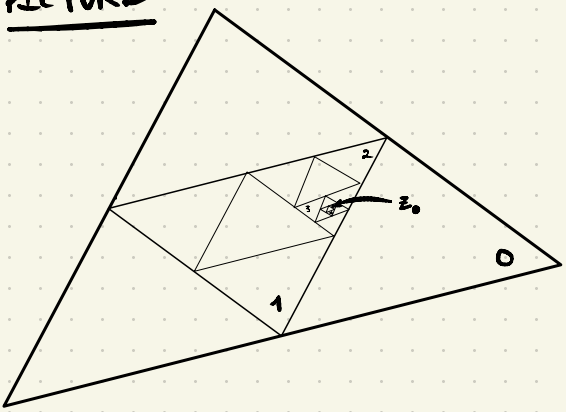
$$\tfrac{1}{2^n} D(\partial T_0) = D(\partial T_n)$$

(**)

NOTE THE T_n ARE CLOSED, NESTED, SHRINKING.

SO HAVE $\bigcap_{n=0}^{\infty} T_n = \{z_0\}$ z_0 SOME POINT of T

PICTURE



WITH z_0 IN HAND WE RECALL LEMMA GIVING $\rho : U \to \mathbb{C}$ CONTINUOUS SO THAT $\rho(z_0) = 0$ AND

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \underbrace{\rho(z)}_{\text{THE "ERROR"}}(z - z_0).$$

FIX $\varepsilon > 0$. SO HAVE $\delta > 0$ WHERE, IF $w \in B(z_0, \delta)$ THEN $|\rho(w)| < \varepsilon$. FIX $n > 0$ SO $T_n \subset B(z_0, \delta)$.

NOW COMPUTE:

$$I(\partial T_n) = \int_{\partial T_n} f \, dz$$

$$= \int_{\partial T_n} f(z_0) + f'(z_0)(z - z_0) \, dz + \int_{\partial T_n} \rho(z)(z - z_0) \, dz \quad \text{(first integral} \to 0 \text{ BY EXERCISE)}$$

$$= \int_{\partial T_n} \rho(z)(z - z_0) \, dz$$

NOTE $|\rho(z) \cdot (z - z_0)| \leq \varepsilon \cdot D(\partial T_n)$.

SO:

$$\frac{1}{4^n} |I(\partial T_0)| \overset{(*)}{\leq} |I(\partial T_n)| \overset{ML}{\leq} \varepsilon \cdot D(\partial T_n) \cdot L(\partial T_n) \overset{(**)}{=} \frac{\varepsilon}{4^n} D(\partial T_0) L(\partial T_0)$$

SO: $|I(\partial T_0)| \leq \varepsilon \cdot D(\partial T_0) \cdot L(\partial T_0)$

THUS $I(\partial T_0) = 0$ $\square$