2026-10-20 LECTURE 7 MA3B8 SAUL SCHLEIMER

LAST TIME:

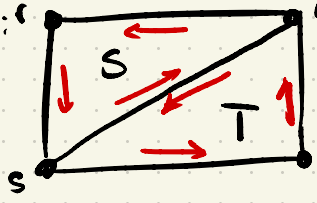
GOURSAT: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ HOLOMORPHIC. SUPPOSE $T \subset U$ TRIANGLE. THEN $\int_{\partial T} f\, dz = 0$

THAT IS: "HOLOMORPHIC FUNCTIONS INTEGRATE TO ZERO ABOUT TRIANGLES".

(1) RECTANGLES:

EXERCISE: U DOMAIN. $f: U \to \mathbb{C}$ HOLO. $R \subset U$ RECTANGLE. THEN $\int_{\partial R} f\, dz = 0$.

PROOF BY PICTURE:



$T = T(p,q,r)$
$S = T(r,s,q)$

DEFINE $I(\gamma) = \int_\gamma f\, dz$.

SO: $I(\partial R) = I([p,q] \cup [q,r] \cup [r,s] \cup [s,p])$

$= I([p,q]) + I([q,r]) + I([r,s]) + I([s,p])$

$= I([p,q]) + I([q,s]) + I([s,p])$

$+ I([r,s]) - I([q,s]) + I([q,r])$

$= I([p,q] \cup [q,s] \cup [s,p])$

$+ I([r,s] \cup [s,q] \cup [q,r])$

$= I(\partial T) + I(\partial S) = 0 + 0 = 0$ $\square$

SO "HOLOMORPHIC FUNCTIONS INTEGRATE TO ZERO ABOUT RECTANGLES".

(2) POLYGONS

DEF: SUPPOSE $p, q \in \mathbb{C}$. SUPPOSE $a, b \in \mathbb{R}$, $a < b$. SUPPOSE $\gamma : [a,b] \longrightarrow \mathbb{C}$ DEF BY

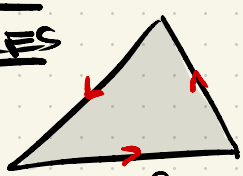
$$\gamma(t) = p \cdot \left(\frac{b-t}{b-a}\right) + q\left(\frac{t-a}{b-a}\right)$$

CALL γ THE AFFINE MAP FROM $[a,b]$ TO $[p,q]$.

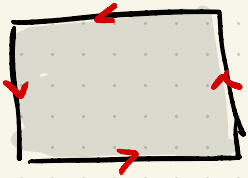
DEF: IF $\gamma : [a,b] \longrightarrow \mathbb{C}$ IS PIECEWISE AFFINE WE CALL IT PIECEWISE LINEAR. [SORRY!].

DEF: SUPPOSE U A DOMAIN. SUPPOSE $P \subset U$ IS COMPACT, NONEMPTY, P° (= INTERIOR P) IS DENSE IN P, P CONNECTED, ∂P A SIMPLE CLOSED PIECEWISE LINEAR CONTOUR THEN CALL P A POLYGON.

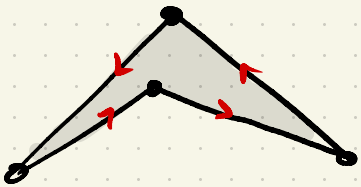
PICTURES



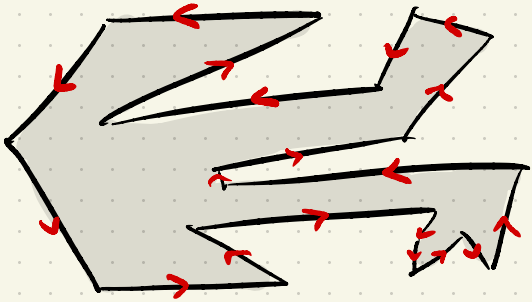
TRIANGLE



RECTANGLE



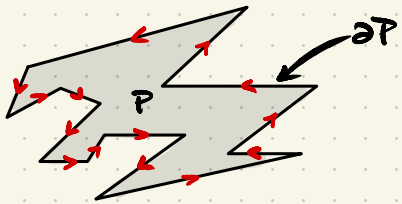
QUADRILATERAL



GENERAL.

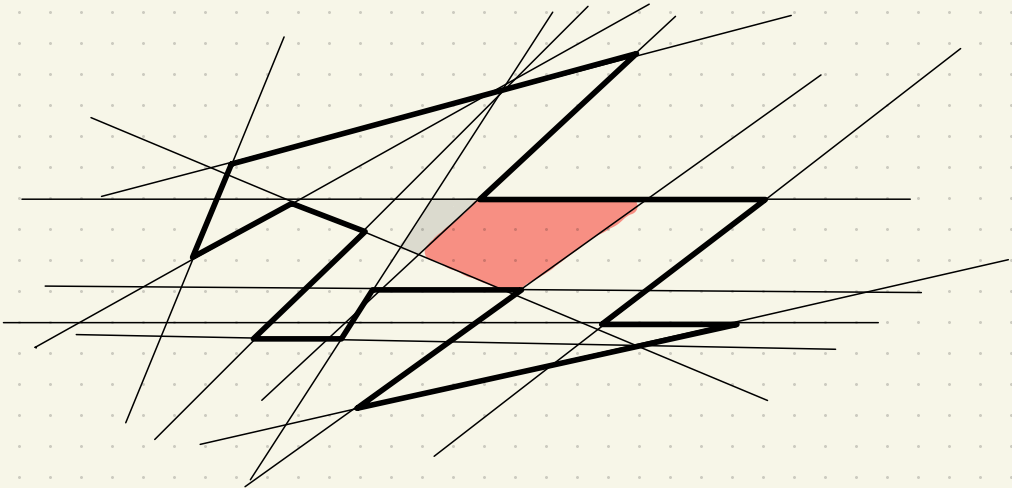
NOTE: NO HOLES (YET!)

CONVENTION: WE ORIENT ∂P SO P IS LOCALLY TO THE **LEFT** OF ∂P.

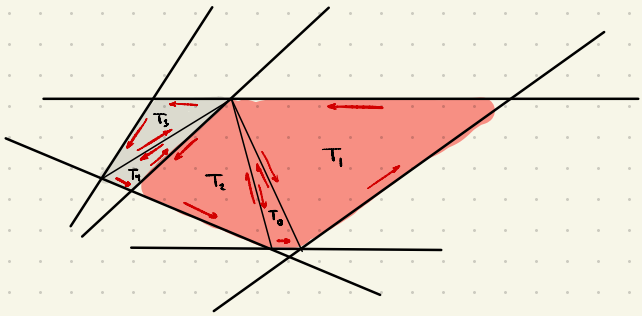


PROP: SUPPOSE U DOMAIN, $f: U \to \mathbb{C}$ HOLOMORPHIC SUPPOSE $P \subset U$ POLYGON. **THEN** $\int_{\partial P} f\,dz = 0$.

PROOF: EXTEND ALL SEGMENTS OF ∂P TO LINES



CUT P USING ALL THESE LINES. THE COMPONENTS REMAINING ARE CONVEX SO CAN BE DIVIDED INTO TRIANGLES MEETING NICELY ALONG SIDES:



THE ORIENTATIONS of THE TRIANGLES T_k ARE ALL POSITIVE SO

(1) $\int_{\partial T_k} f\,dz = 0$ FOR ALL k

(2) $\int_{\partial P} f\,dz = \sum_k{}' \int_{\partial T_k} f\,dz$. } INTERNAL INTEGRALS CANCEL IN PAIRS.

SO $\int_{\partial P} f\,dz = 0$. □

SO: "HOLOMORPHIC FUNCTIONS INTEGRATE TO ZERO ABOUT POLYGONS"