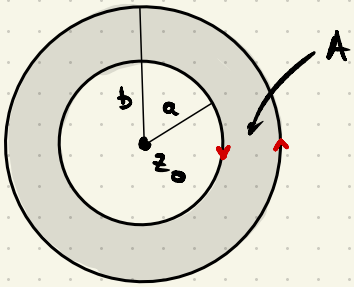
2026-10-21 LECTURE 8 MA3B8 SAUL SCHLEIMER

① ANNULI:

DEFINE SUPPOSE $z_0 \in \mathbb{C}$, $0 < a < b$ IN $\mathbb{R}$.

$$A(z_0; a, b) = \{ w \in \mathbb{C} \mid a < |z_0 - w| < b \}$$

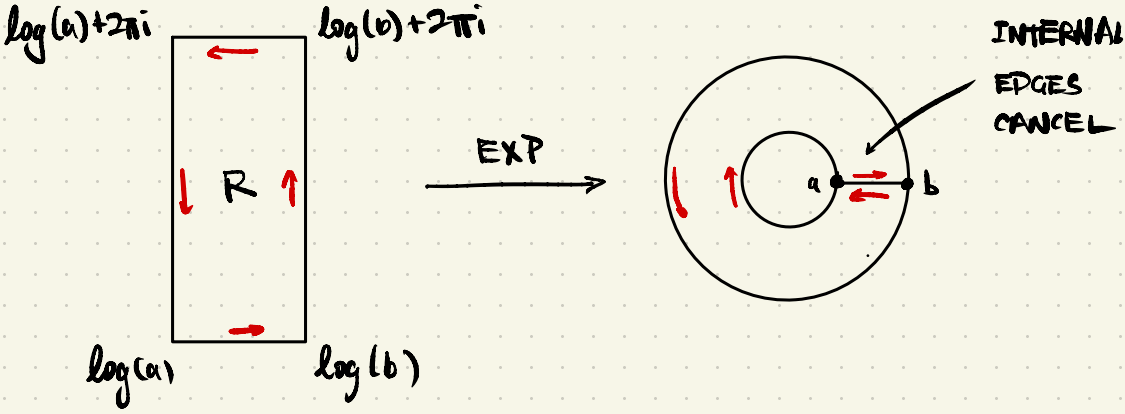
PICTURE



NOTE $\partial A = C(z_0; a) \sqcup C(z_0; b)$ IS THE UNION of TWO CIRCLES. WE ORIENT THESE SO A IS IS ON THEIR LEFT. SO OUTER BOUNDARY IS ORIENTED ANTI-CLOCKWISE WHILE INNER BOUNDARY IS ORIENTED CLOCKWISE.

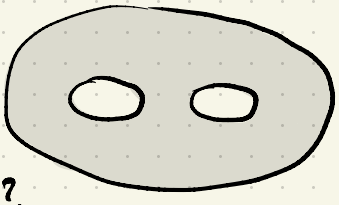
LEMMA: SUPPOSE U DOMAIN, $f : U \to \mathbb{C}$ HOLOMORPHIC, $A \subset U$ ANNULUS. THEN $\int_{\partial A} f \, dz = 0$

PROOF [TOO CLEVER BY HALF] SUPPOSE $z_0 = 0$. CONSIDER R THE RECTANGLE WITH VERTICES



SO $\int_{\partial A} f\,dz = \int_{\partial R} \underbrace{(f\circ EXP)\cdot EXP'(w)}_{\text{HOLOMORPHIC}}\,dw = 0$. $\square$

SO WE NEEDED TO PARAMETRISE A IN A CLEVER, NOT VERY GENERAL, WAY. FOR EXAMPLE HERE IS A DISC MINUS TWO SMALLER DISCS. WHAT IS THE VERSION of EXP FOR THIS REGION?.



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(2) PRIMITIVES: SUPPOSE U IS A DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ IS A FUNCTION. SUPPOSE $F: U \to \mathbb{C}$ IS HOLOMORPHIC AND $F' = f$. THEN WE SAY "F IS A PRIMITIVE FOR f" AND "f HAS A PRIMITIVE".

NOTE: THESE ARE ANTIDERIVATIVES WITH A NEW NAME.

NOTE: SUPPOSE F, G ARE PRIMITIVES FOR f. THEN $F - G$ IS CONSTANT.

PROOF $(F-G)' = F' - G' = f - f \equiv 0$ AND MVT $\square$

LEMMA: SUPPOSE $B = B(w_0; r)$. SUPPOSE $f: B \to \mathbb{C}$ CONTINUOUS. SUPPOSE $\int_{\partial T} f\,dz = 0$ FOR ALL TRIANGLES $T \subset B$.

THEN f HAS A PRIMITIVE IN B.

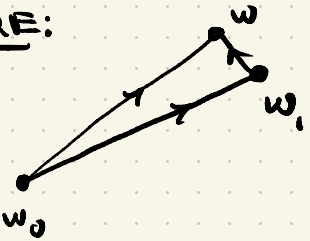
PROOF: FIX $w \in B$. DEFINE $F(w) = \int_{[w_0, w]} f\,dz$.

SO $F: B \to \mathbb{C}$ FUNCTION.

FIX $w_1 \in B$.

CLAIM: $F'(w_1)$ EXISTS AND EQUALS $f(w_1)$.

PICTURE:



SO $\int_{[w_0,w]} + \int_{[w,w_1]} + \int_{[w_1,w_0]} = 0$

NOTE f CONTINUOUS AT w_1.

SO FIX $\varepsilon > 0$. SO HAVE $\delta > 0$

WITH $|z - w_1| < \delta$ IMPLIES $|f(z) - f(w_1)| < \varepsilon$.

WE NOW COMPUTE (ASSUMING $|w - w_1| < \delta$)

$$\left| \frac{F(w) - F(w_1)}{w - w_1} - f(w_1) \right| = \frac{|F(w) - F(w_1) - f(w_1)(w - w_1)|}{|w - w_1|}$$

$$= \frac{\left| \int_{[w_0,w]} f\,dz - \int_{[w_0,w_1]} f\,dz - \int_{[w_1,w]} f(w_1)\,dz \right|}{|w - w_1|}$$

$$= \frac{\left| \int_{[w_1,w]} f\,dz - \int_{[w_1,w]} f(w_1)\,dz \right|}{|w - w_1|} = \frac{\left| \int_{[w_1,w]} (f(z) - f(w_1))\,dz \right|}{|w - w_1|}$$

$$\underset{\text{ML}}{\leq} \frac{\varepsilon \cdot |w - w_1|}{|w - w_1|} = \varepsilon \qquad \square$$
