2026-10-23 LECTURE 9 MA3B8 SAUL SCHLEIMER

LAST TIME:

LEMMA: SUPPOSE $B = B(z_0; R)$. SUPPOSE $f: B \to \mathbb{C}$ CONTINUOUS. SUPPOSE $\int_{\partial T} f\, dz = 0$ FOR ALL TRIANGLES $T \subset B$. THEN f HAS A PRIMITIVE IN B.

COROLLARY: SUPPOSE $B = B(z_0; r)$. SUPPOSE $f: B \to \mathbb{C}$ HOLOMORPHIC. THEN f HAS A PRIMITIVE IN B.

CAUTION: $f(z) = \frac{1}{z}$ HOLOMORPHIC IN $\mathbb{C}^\times = \mathbb{C} - \{0\}$. SO f HAS PRIMITIVES IN DISCS IN $\mathbb{C}^\times$. BUT f DOES NOT HAVE A PRIMITIVE IN $\mathbb{C}^\times$.

PROPOSITION: SUPPOSE U DOMAIN. SUPPOSE $f: U \to \mathbb{C}$ CTS. THEN (a) f HAS A PRIMITIVE IN U [NOT JUST IN A DISC!]

IFF

(b) $\int_\gamma f\, dz = 0$ FOR ALL CLOSED CONTOUR $\gamma: [a,b] \to U$

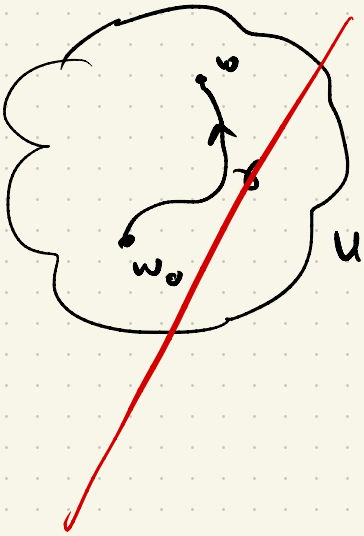
PROOF: SUPPOSE $F: U \to \mathbb{C}$ A PRIMITIVE FOR f. THEN

$$\int_\gamma f\, dz = \int_\gamma F'\, dz = \int_a^b F'(\gamma(t)) \cdot \gamma'(t)\, dt$$

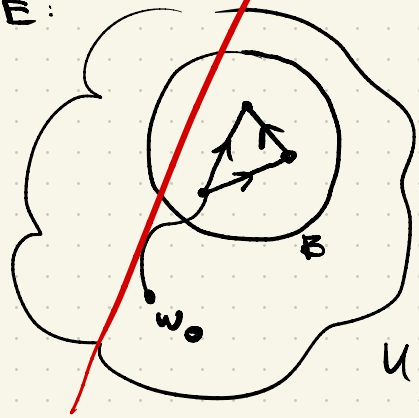
$$= F(\gamma(b)) - F(\gamma(a)) = 0.$$

FOR $\gamma: [a,b] \to U$ CLOSED.

SUPPOSE INSTEAD WE HAVE (b). THEN FIX $w_0 \in U$. DEFINE $F(w) = \int_\gamma f\, dz$ FOR ANY CONTOUR $\gamma: [a,b] \to U$ SO THAT $\gamma(a) = w_0$, $\gamma(b) = w$. SKIP



SO $F(w)$ WELL-DEF BY HYPOTHESIS ON f. WE CHECK $F' = f$ AS BEFORE:



SKIP

□

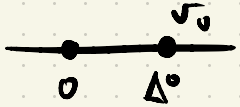
② SINGULAR SIMPLICIES

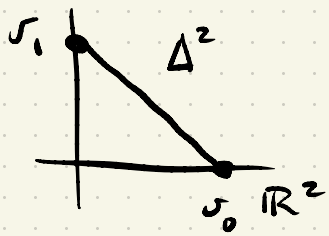
WE GIVE A "FORMALISM" FOR ADDING CONTOURS.

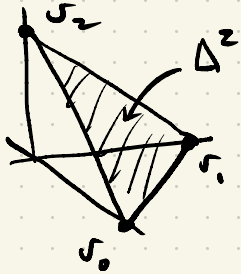
DEF: $\Delta^n = \{ x \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1,\ x_i \geq 0 \}$

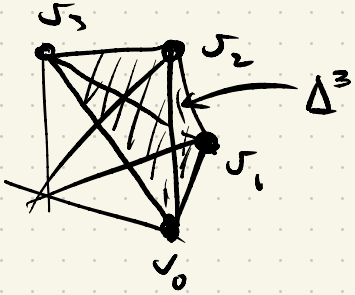
THE STANDARD n-SIMPLEX

PICTURES









LET $v_k \in \Delta^n$ BE THE k^{th} VERTEX

$v_k = (0, 0, \cdots, 0, 1, 0, \cdots 0)$

(k^{th} COORD)

DEF: SUPPOSE X IS A TOP. SPACE. SUPPOSE $\sigma^n : \Delta^n \to X$ IS CONTINUOUS. CALL σ^n A SINGULAR n-SIMPLEX.

DEF: $C_n(X) = \left\{ \sum_k a_k \sigma_k^n \;\middle|\; \text{FINITE } \mathbb{Z}\text{-LINEAR SUMS of SING. } n\text{-SIMPLICES} \right\}$

THE GROUP of n-CHAINS.

[= FREE ABELIAN GROUP GEN. BY SING n-SIMPLICES

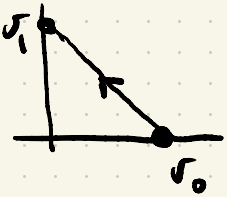
EXAMPLE: WE CAN REPARAM ANY CONTOUR

$$\Delta^1 \longrightarrow [a,b] \xrightarrow{\gamma} \mathbb{C}$$

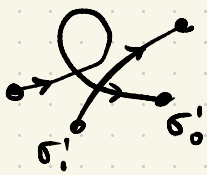
TO GET A SING. 1-SIMPLEX. USE:

$$(\lambda_0, \lambda_1) \longmapsto a\lambda_0 + b\lambda_1 \longmapsto \gamma(a\lambda_0 + b\lambda_1).$$

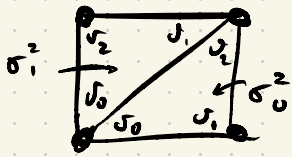
NOTE ORIENTATION



PICTURES



ONE-CHAIN



TWO-CHAIN.

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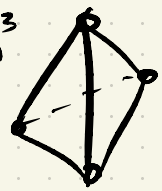
③ FACES: FIX k $0 \le k \le n$. DEFINE THE k^{TH} FACE of Δ^n BY

$$\Delta^{n-1} \longrightarrow \Delta^n$$
$$(\lambda_i) \longmapsto \sum_{i<k} \lambda_i v_i + \sum_{i \ge k} \lambda_i v_{i+1}$$

PICTURE Δ^0 HAS NO FACES (!)

Δ^1  HAS TWO

Δ^2  HAS THREE

Δ^3  HAS FOUR FACES.

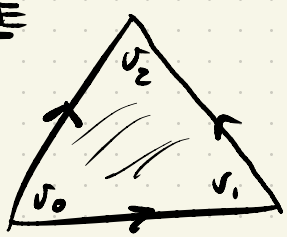
Δ^n HAS $(n+1)$-FACES.

DENOTE THE ABOVE FACE BY $[v_0, v_1 \cdots v_{k-1}, \hat{v}_k, v_{k+1} \cdots v_n]$

SO WE OMIT v_k.

FACES of Δ^2 ARE $[v_1, v_2]$, $[v_0, v_2]$, $[v_0, v_1]$

PICTURE



DOES NOT OBEY ORIENTATION CONVENTION... SO NEED FOLLOWING:

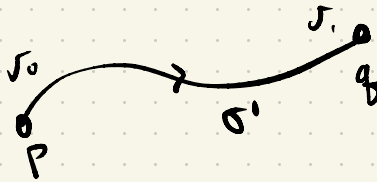
DEF: SUPPOSE $\sigma^n : \Delta^n \longrightarrow X$ SING. SIMPLEX.

THEN SET $\partial_i \sigma^n = \sum_i (-1)^k \sigma^n | [v_0, \cdots \hat{v}_k, \cdots v_n]$

EXTEND LINEARLY TO GET

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X).$$

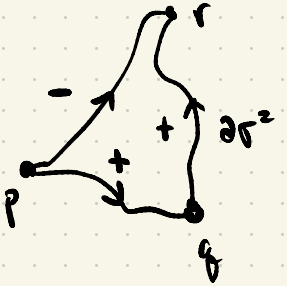
EXAMPLE: $\partial \sigma^1 = \sigma^1|[v_1] - \sigma^1|[v_0]$



$\partial_1 \sigma^1 = p - q.$



$$\partial_2 \sigma^2 = \sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]$$



(4) CHAINS, CYCLES, BOUNDARIES:

SO: $C_n(X)$ IS THE ABELIAN GROUP of n-CHAINS.

AND $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ IS THE BOUNDARY OPERATOR.

DEF: $c \in C_n(X)$ IS AN n-CYCLE IF $\partial_n c = 0$.

DEF: $c \in C_n(X)$ IS AN n-BOUNDARY IF THERE IS SOME $d \in C_{n+1}(X)$ WITH $\partial_{n+1} d = c$.