MA3B8: COMPLEX ANALYSIS

SAUL SCHLEIMER

0. Introduction

> *...the shortest and best way between two truths of the real domain often passes through the imaginary one.*
>
> – Jacques Hadamard
> *The psychology of invention in the mathematical field*

These notes set out the "material of record" for the 2026-2027 module MA3B8 (complex analysis) at Warwick. We give solutions to all exercises (except one) in Appendix A (starting on page 53). We also give a short index of notation (on page 69) and a list of references (on page 70).

If you find any errors in the notes, or any possible improvements, please inform me directly (at s.schleimer@warwick.ac.uk) or use the anonymous form:

`https://forms.gle/AnXkWf5DM8epZgF8A`

Acknowledgements. I first learnt this remarkable subject when taking courses from Donald Sarason and Curt McMullen at UC Berkeley. Sarason's book [13] gives a wonderful exposition of the topic for advanced undergraduates. I have also spent many happy hours reading Ahlfors' classic text [1].

I have also benefited directly from many enlightening conversations with Ed Brambley, Jeremy Kahn, Diane Maclagan, Cagri Sert, and Peter Topping.

Finally, I am deeply grateful to the many students who have asked me questions and pointed out mistakes; I dedicate these notes to you.

AI declaration. I wrote the first version of the notes, for the 2025-26 academic year, without any AI. This year I used various AIs as search engines, as line-by-line copyeditors, and as occasional conversational partners: replacements for my human colleagues, after they tired of hearing me go on and on about Cauchy's theorem. In the course of editing, one of the AIs pointed out a subtle mistake in a previous version of Proposition 9.1: the definition of midpoint subdivision (of chains) had an orientation-plus-sign error. In addition, an AI suggested the surprisingly (to me) simple proof of Claim 12.2 after I gave up on finding an argument using only ruler-and-compass.

All of the text was written, and all of the figures drawn, by me. Any remaining errors are mine – please let me know of any you find.

1. Rectangular and polar coordinates

1.1. **Notation.** We use the following:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the natural numbers.
- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are, respectively, the integers, rationals, and reals.
- $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ is the complex numbers.

We assume the usual algebraic properties of each of these.

We assume the usual topological and geometric properties of the plane; this includes the idea of homeomorphism of subsets. When convenient we will identify $\mathbb{R}$ with either the x–axis in $\mathbb{R}^2$ or the real axis in $\mathbb{C}$.

We assume that the usual real-valued functions (cosine, sine, exponential, logarithm, square-root) with their usual properties are available. We also assume the usual analytic properties of real-valued functions of several variables, giving references where needed.

1.2. **Rectangular coordinates.** Suppose that

$$z = x + iy$$

is a complex number. Then we define

$$\textsc{Real}(z) = x \quad \text{and} \quad \textsc{Imag}(z) = y$$

to be the *real* and *imaginary* parts of z. We also define

$$\bar{z} = x - iy$$

to be the *complex conjugate* of z. We define

$$|z| = \sqrt{x^2 + y^2}$$

to be the *magnitude* of z. Note that $z\bar{z} = |z|^2$. Also, $|z| = 0$ if and only if $z = 0$.

Suppose that $r > 0$ is a positive real number. We define

$$B(z; r) = \{w \in \mathbb{C} : |w - z| < r\}$$

and

$$\overline{B(z; r)} = \{w \in \mathbb{C} : |w - z| \leq r\}$$

Thus $B(z; r)$ is the open, and $\overline{B(z; r)}$ is the closed, *disc* of radius r about z. (These are also called the open and closed *balls* of radius r about z.) We call $B^\times(z; r) = B(z; r) - \{z\}$ the *punctured* open disc of radius r about z. Similarly we call $\mathbb{C}^\times = \mathbb{C} - \{0\}$ the *punctured* plane.

Suppose that P is a non-empty subset of the plane $\mathbb{C}$. The *interior* of P, denoted P°, consists of those points $z \in P$ so that there is some $r > 0$ with $B(z; r) \subset P$. The *closure* of P, denoted $\overline{P}$, is the complement of the interior of $\mathbb{C} - P$. Said another way, the interior of P is the largest open subset of P while the closure of P is the smallest closed superset of P. The *topological boundary of* P is $\partial P = \overline{P} - P^\circ$: the closure of P minus the interior of P.

We define

$$C(z;r) = \{w \in \mathbb{C} : |w - z| = r\}$$

Thus $C(z;r)$ is the *circle* of radius r about z. Note that $C(z;r)$ is the topological boundary of both $B(z;r)$ and $\overline{B(z;r)}$. As a bit of notation we take $\mathbb{D} = B(0;1)$ and $S^1 = C(0;1)$; these are the *(open) unit disc* and the *unit circle*. We also use $\mathbb{D}^\times = \mathbb{D} - \{0\}$ for the *punctured unit disc*. We also define $\mathbb{H} = \{z \in \mathbb{C} \mid \text{IMAG}(z) > 0\}$: the *upper-half plane*.

Suppose that Z is a subset of the plane. We define

$$N(Z;r) = \{w \in \mathbb{C} : |w - z| < r \text{ for some } z \in Z\}$$

to be the *open r–neighbourhood* of Z. So $N(Z;r)$ is the union of the open r–discs about points z of Z.

1.3. **Polar coordinates.** Suppose that $z = x+iy \neq 0$. Thus $x^2+y^2 > 0$. So we may form

$$C = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad S = \frac{y}{\sqrt{x^2+y^2}}$$

Note that $C^2 + S^2 = 1$. We say that $\theta \in \mathbb{R}$ is an *argument* of z if

$$\cos(\theta) = C \quad \text{and} \quad \sin(\theta) = S$$

We define $\text{ARG}(z)$ to be the set of arguments of z.

Exercise 1.4. Suppose that $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Prove that $\text{ARG}(z)$ is non-empty. Prove that, for any argument θ, we have $\text{ARG}(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$. ◇

Thus the argument is a *multi-valued function*: a one-to-many relation that is defined everywhere on its domain. This subtlety lies at the heart of the theory of Riemann surfaces.

Immediately from the definitions we deduce that, for any $z \in \mathbb{C}^\times$ and for any $\theta \in \text{ARG}(z)$, there is a *polar form*:

$$z = |z|(\cos(\theta) + i\sin(\theta))$$

As a useful bit of notation we write $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. The addition law implies that $e^{i\theta+i\eta} = e^{i\theta}e^{i\eta}$.

Definition 1.5. Fix a positive integer n. Set $\omega_n = e^{2\pi i/n}$. Then ω_n and its powers are the nth *roots of unity*. ◇

By the addition law, we have $\omega_n^n = 1$. Thus, under multiplication, the nth roots of unity form a cyclic group of order n.

Exercise 1.6. Suppose that $z, w \in \mathbb{C}^\times$. Prove that

$$|zw| = |z||w| \qquad \text{and} \qquad \text{ARG}(zw) = \text{ARG}(z) + \text{ARG}(w) \qquad ◇$$

Exercise 1.7. Suppose that $z, w \in \mathbb{C}^\times$. Suppose that θ and η are arguments of z and w, respectively. Prove the cosine law

$$|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta)$$

directly from the definitions and the addition law. ◇

With arguments and the cosine law in hand, we can identify the geometry of $\mathbb{C}$ with that of $\mathbb{R}^2$.

Definition 1.8. Suppose that a and b are complex numbers, with $a \neq 0$. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto az + b$. Then we call f an *(orientation preserving) similarity*. If $|a| = 1$, then f is an *(orientation preserving) congruence*. If $a = 1$, then f is a *translation*. If $b = 0$, then f is a *homothety*[1] about the origin. If a is real and positive, and $b = 0$, then f is a *dilation* about the origin. If $|a| = 1$ and $b = 0$, then f is a *rotation* about the origin. ◇

Exercise 1.9. Show that $\textsc{Sim}(\mathbb{C})$, the set of similarities of $\mathbb{C}$ together with the operation of function composition, is a group. Show that each subclass (congruence, translation, homothety, dilation, and rotation) gives a subgroup. ◇

Suppose that U and V are subsets of the plane. We say that U and V are *similar* if there is a similarity sending U to V. We make the same definition for the various subclasses in Definition 1.8.

Exercise 1.10. Suppose that $z, w \in \mathbb{C}$. Suppose that $z \neq 0$ and that $|z - w| < |z|$. Prove that $w \neq 0$. Prove, furthermore, that for any $\theta \in \textsc{Arg}(z)$ there is a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi/2$. ◇

We now prove that "the" argument is "continuous" on the punctured plane $\mathbb{C}^\times$.

Lemma 1.11. *Suppose that $z \in \mathbb{C}^\times$. Suppose that $\epsilon \in (0, \pi/2)$. Then there is a $\delta > 0$ with the following property.*

Suppose that $w \in \mathbb{C}$ has $|z - w| < \delta$. Then $w \neq 0$ and, for every $\theta \in \textit{Arg}(z)$, there is a unique $\eta \in \textit{Arg}(w)$ so that $|\theta - \eta| < \epsilon$.

Proof. We take $\delta = \sin(\epsilon)|z|$. Thus δ is positive and less than $|z|$. So $|z - w| < \delta < |z|$. By Exercise 1.10 we have $w \neq 0$ and we have a unique $\eta \in \textsc{Arg}(w)$ so that $|\theta - \eta| < \pi/2$.

Since $|z - w|^2 < \delta^2 = |z|^2 \sin^2(\epsilon)$ the cosine law gives us

$$|z|^2 + |w|^2 - 2|z||w| \cos(\theta - \eta) < |z|^2 \sin^2(\epsilon)$$

Moving terms and simplifying gives

$$|z|^2 \cos^2(\epsilon) + |w|^2 < 2|z||w| \cos(\theta - \eta)$$

Subtracting from both sides gives

$$|z|^2 \cos^2(\epsilon) - 2|z||w| \cos(\epsilon) + |w|^2 < 2|z||w| \cos(\theta - \eta) - 2|z||w| \cos(\epsilon)$$

Thus

$$(|z| \cos(\epsilon) - |w|)^2 < 2|z||w|(\cos(\theta - \eta) - \cos(\epsilon))$$

[1] That is, scaling-and-rotation. A clearer (but longer) name would be *complex homothety*.

The left-hand side is non-negative. Replacing it by zero and simplifying gives

$$\cos(\epsilon) < \cos(\theta - \eta)$$

Since cosine is even, and decreasing on $[0, \pi]$, we deduce that $|\theta - \eta| < \epsilon$, as desired. □

2. Contour integrals

2.1. **Domains.**

Definition 2.2. Suppose that $U \subset \mathbb{C}$ is non-empty, open, and connected. Then we call U a *domain*. ◇

As suggested by Figure 2.3, domains can be quite wild.

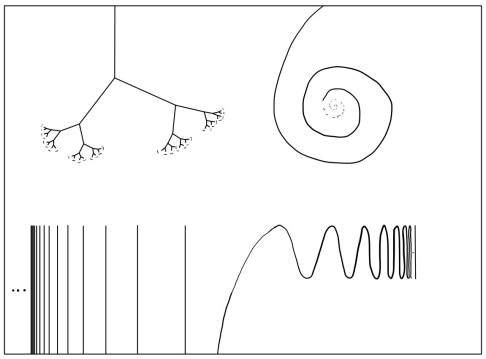

Figure 2.3. A wild domain. Here U is the open rectangle minus four closed sets: a *dendrite* (upper left), a *spiral* (upper right), a *comb* (lower left), and a *topologist's sine curve* (lower right).

Nonetheless, domains have several nice properties.

Exercise 2.4. Domains are path-connected. In fact, they are path-connected by *axis-aligned paths*: paths that

- alternate between running parallel to the real and imaginary axes and
- have finitely many corners. ◇

Exercise 2.5. Suppose that U is a domain. Suppose that $z_0 \in U$. Prove that $U^\times = U - \{z_0\}$ is a domain. ◇

Exercise 2.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $P \subset U$ is compact and non-empty. Prove that there is a constant $\epsilon > 0$ so that the open neighbourhood $N(P; \epsilon)$ is contained in U. ◇

2.7. **Branches.** With domains in hand, we give one solution to the "problem" of multi-valued functions.

Definition 2.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that F is a multi-valued function on U. A *branch* of F is

- a domain $V \subset U$ and
- a continuous function $f\colon V \to \mathbb{C}$

so that, for all $z \in V$, we have $f(z) \in F(z)$. ◇

For example, ARG has a branch $\arg\colon \mathbb{H} \to \mathbb{R}$ obtained by always picking $\arg(z) \in (0, \pi)$. Here, the continuity of arg follows from Lemma 1.11.

Definition 2.9. We define a multi-valued function on $\mathbb{C}^\times$ as follows:

$$\text{SQRT}(z) = \{w \in \mathbb{C}^\times \mid w^2 = z\}$$

We call the elements of $\text{SQRT}(z)$ the *square roots* of z. ◇

Exercise 2.10. Prove that any $z \in \mathbb{C}^\times$ has exactly two square roots.

Suppose that $B = B(z_0; r_0)$ is a disc with $0 < r_0 < |z_0|$. Prove that SQRT has exactly two branches with domain equal to B. ◇

We return to this topic in Section **??** where we construct holomorphic branches of the logarithm.

2.11. **Contours.**

I keep a close watch on this heart of mine
I keep my eyes wide open all the time
I keep the ends out for the tie that binds
Because you're mine, I walk the line

– Johnny Cash
I walk the line

Definition 2.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $[a, b]$ is a closed non-trivial interval in $\mathbb{R}$.[2] A *contour in* U is a piecewise C^1 function $\gamma\colon [a, b] \to U$. That is, we have a finite sequence

$$(a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b) \subset [a, b]$$

so that, for all k, the restriction $\gamma_k = \gamma|[t_k, t_{k+1}]$ is continuously differentiable. We call $\gamma_k = \gamma|[t_k, t_{k+1}]$ a *piece* of γ. We call the point t_k a *breakpoint* and its image $\gamma(t_k)$ a *vertex* of γ. ◇

We do not require the left- and right-hand derivatives to agree at breakpoints.

Some authors (for example [14, page 19]) additionally assume that all contours have non-vanishing derivative. We do not; instead we add the hypothesis once (to Definition 11.1) and use it once (in Lemma 11.3).

[2] It will be convenient to also allow *trivial contours*, where $a = b$. A trivial contour is a bit like the number zero: absurd, but it makes everything more tidy.

Example 2.13. Take $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$, a third root of unity. We consider the contour $\gamma\colon [0,3] \to \mathbb{C}$ given by:

$$\gamma(t) = \begin{cases} (1-t)\cdot 1 + (t-0)\cdot \omega, & 0 \le t \le 1 \\ (2-t)\cdot \omega + (t-1)\cdot \omega^2, & 1 \le t \le 2 \\ (3-t)\cdot \omega^2 + (t-2)\cdot 1, & 2 \le t \le 3 \end{cases}$$

This contour traces the boundary of an equilateral triangle with vertices at the third roots of unity. So it has three pieces, its breakpoints are at the integers, and its vertices are at the roots of unity. Note that the right and left derivative disagree at the breakpoints. ◇

Suppose that $\gamma\colon [a,b] \to U$ is a contour. We say that γ is *closed* if $\gamma(b) = \gamma(a)$. (This is not the same as the topological notion of a closed set.) We say that γ is *simple* if $\gamma(r) = \gamma(s)$ implies that $r = s$ or $\{r,s\} = \{a,b\}$. (That is, simple contours are either injective, or are closed and fail to be injective only at their endpoints.) The triangular contour of Example 2.13 is both simple and closed.

Example 2.14. Suppose that $\gamma\colon [0,b] \to \mathbb{C}$ is given by $\gamma(\theta) = e^{i\theta}$.

- If $b = \pi$, then γ is simple but is not closed.
- If $b = 2\pi$, then γ is simple and is closed.
- If $b = 3\pi$, then γ is not simple and is not closed.
- If $b = 4\pi$, then γ is not simple but is closed.

In each case the contour has only one piece. ◇

2.15. **Integrals: definition and examples.**

Definition 2.16. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We define the *contour integral* of f along γ as follows.

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

If γ has multiple pieces then we break the right-hand side into the corresponding finite sum of integrals. ◇

Here is our first, and most important, example of a contour integral.

Example 2.17. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma\colon [0,2\pi] \to U$ is given by $\gamma(\theta) = e^{i\theta}$. Suppose that $f\colon U \to \mathbb{C}$ is given by $f(z) = 1/z$. Then we have:

$$\int_\gamma f\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i \qquad \diamond$$

In Example 2.17 the contour "winds" once (positively) about the origin. (We give a precise definition of "winds" in Definition 3.7.) And, as it happens, the integrand has a *pole* – a particular kind of singularity

– at the origin. (We give a precise definition of "pole" in Section **??**.) If we change the nature of the pole then the answer changes.

Exercise 2.18. Suppose that U and γ are as in Example 2.17. Suppose that k is an integer. Suppose that $f_k \colon U \to \mathbb{C}$ is given by $f_k(z) = z^k$. Compute $I_k = \int_\gamma f_k \, dz$. ◇

Moving the contour and integrand by the same amount does not affect the integral. More interestingly, in this particular case, shrinking the contour also does not affect the integral.

Example 2.19. Suppose that w is a point of $\mathbb{C}$. Suppose that $U = \mathbb{C} - \{w\}$ is a punctured plane. Suppose that $r > 0$ is real. Suppose that $\gamma \colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = w + re^{i\theta}$. So $\gamma'(\theta) = ire^{i\theta}$. Suppose that $f \colon U \to \mathbb{C}$ is given by $f(z) = 1/(z - w)$. Then we have:

$$\int_\gamma f \, dz = \int_\gamma \frac{dz}{z - w} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} \, d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

If $k \neq -1$, and if we replace f by $(z - w)^k$, then as in Exercise 2.18 the integral vanishes. ◇

If we move the contour so that it does not wind about the origin, but leave the integrand unchanged, then the integral again vanishes. However, the proof is more delicate.

Exercise 2.20. Suppose that $U = \mathbb{C}^\times$ is the punctured plane. Suppose that $\gamma \colon [0, 2\pi] \to U$ is given by $\gamma(\theta) = 2 + e^{i\theta}$. Suppose that $f \colon U \to \mathbb{C}$ is given by $f(z) = 1/z$. Give a direct proof that $\int_\gamma f \, dz = 0$. ◇

All of the above are shadows of one of the main results of this module: Cauchy's theorem (Theorem 14.11).

2.21. **Integrals: linearities, deformations, and orientations.** The contour integral is linear with respect to the integrand, as follows.

Lemma 2.22. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that $g \colon U \to \mathbb{C}$ is a continuous function. Suppose that $p, q \in \mathbb{C}$ are complex numbers. Then we have:*

$$\int_\gamma (p \cdot f + q \cdot g) \, dz = p \int_\gamma f \, dz + q \int_\gamma g \, dz \qquad \square$$

This has several applications.

Exercise 2.23. Suppose that U and γ are as in Example 2.17. Suppose that f is a *Laurent polynomial over* $\mathbb{C}$: a finite sum $f = \sum_{k=m}^{n} a_k z^k$ where $m < 0 < n$ and all of the a_k are complex. Prove the following.

$$\int_\gamma f \, dz = 2\pi i a_{-1} \qquad \diamond$$

We can also understand what happens when we shrink a round contour.

Exercise 2.24. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Fix w in U. Suppose that $\delta > 0$ is real with $C(w;\delta) \subset U$. Define $\gamma\colon [0, 2\pi] \to U$ by $\gamma(\theta) = w + \delta e^{i\theta}$. Now define

$$I_\delta = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w}\, dz$$

Prove, with w fixed, that I_δ tends to $f(w)$ as δ tends to zero. ◇

The contour integral is also linear with respect to the domain of integration, as follows.

Lemma 2.25. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that c is a point of $[a, b]$. Define $\gamma_a = \gamma|[a, c]$ and $\gamma_b = \gamma|[c, b]$. Then we have:*

$$\int_\gamma f\, dz = \int_{\gamma_a} f\, dz + \int_{\gamma_b} f\, dz$$ □

This has an immediate consequence.

Exercise 2.26. Suppose that $\gamma\colon [a, b] \to \mathbb{C}$ is a contour. Then

$$\int_\gamma dz = \gamma(b) - \gamma(a)$$ ◇

The contour integral is (almost) independent of parametrisation, as follows.

Lemma 2.27. *Suppose that U, γ, and f are as in Definition 2.16. Suppose that $\delta\colon [c, d] \to [a, b]$ is piecewise C^1. Suppose that $\gamma \circ \delta$ is again a contour. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have:*

$$\int_{\gamma\circ\delta} f\, dz = \int_\gamma f\, dz$$

If, on the other hand, $\delta(c) = b$ and $\delta(d) = a$, then

$$\int_{\gamma\circ\delta} f\, dz = -\int_\gamma f\, dz$$

In the first case we say that γ and $\gamma \circ \delta$ have the *same orientation*. In the second case we say that they have *opposite orientations*.

Proof of Lemma 2.27. We compute as follows:

$$\begin{aligned}
\int_{\gamma\circ\delta} f\, dz &= \int_c^d f(\gamma(\delta(u)))\, (\gamma \circ \delta)'(u)\, du && \text{Definition 2.16} \\
&= \int_c^d f(\gamma(\delta(u)))\, \gamma'(\delta(u))\, \delta'(u)\, du && \text{chain rule} \\
&= \int_a^b f(\gamma(t))\, \gamma'(t)\, dt && \text{substitution} \\
&= \int_\gamma f\, dz && \text{Definition 2.16}
\end{aligned}$$

If δ reverses the endpoints then we instead obtain the following.

$$\int_{\gamma\circ\delta} f\,dz = \int_b^a f(\gamma(t))\,\gamma'(t)\,dt = -\int_\gamma f\,dz \qquad \square$$

2.28. **Arc-length.** Suppose that Q is a geometric quantity such as arc-length, angle, area, and so on. Then the *element of* Q is the differential we integrate to obtain Q. In Definition 2.16, we have already encountered the *element of (change of) position*:

$$dz = dx + i\,dy$$

The name is explained by Exercise 2.26.

A more substantial example is the *element of arc-length* [3, page 28]:

$$ds = \sqrt{dx^2 + dy^2}$$

We use this as follows.

Definition 2.29. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a contour. We define the *arc-length* of γ as follows.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_a^b |\gamma'(t)|\,dt \qquad \diamondsuit$$

Example 2.30. The upper half of the unit circle is parametrised by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0,\pi]$. Since this has unit speed, we have the following.

$$L(\gamma) = \int_\gamma \sqrt{dx^2 + dy^2} = \int_0^\pi |\gamma'(\theta)|\,d\theta = \int_0^\pi d\theta = \pi$$

The upper half of the circle is also parametrised (with the opposite orientation) by

$$\beta(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

for $t \in [-1,1]$. Computing, we have the following.

$$\begin{aligned}
L(\beta) &= \int_\beta \sqrt{dx^2 + dy^2} = \int_{-1}^1 |\beta'(t)|\,dt \\
&= \int_{-1}^1 2\sqrt{\frac{(1-t^2)^2}{(1+t^2)^4} + \frac{(2t)^2}{(1+t^2)^4}}\,dt \\
&= \int_{-1}^1 \frac{2}{(1+t^2)^2}\sqrt{(1-t^2)^2 + (2t)^2}\,dt \\
&= \int_{-1}^1 \frac{2}{1+t^2}\,dt \qquad \text{so substitute } t = \tan(\theta) \\
&= \int_{-\pi/4}^{\pi/4} 2\,d\theta = \pi \qquad \diamondsuit
\end{aligned}$$

As the above example suggests, the arc-length integral is independent of parametrisation.

Lemma 2.31. *Suppose that U and γ are as in Definition 2.29. Suppose that $\delta\colon [c,d] \to [a,b]$ is piecewise C^1 and bijective. Suppose that $\gamma \circ \delta$ is again a contour. Then we have $L(\gamma \circ \delta) = L(\gamma)$.*

Exercise 2.32. Prove Lemma 2.31. ◇

2.33. **The ML–inequality.**

Definition 2.34. Suppose that U is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $\gamma\colon [a,b] \to U$ is a contour. Then we define:

$$\int_\gamma |f\,dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt$$ ◇

Exercise 2.35. With hypotheses as in Definition 2.34, prove that

$$\left|\int_\gamma f\,dz\right| \leq \int_\gamma |f\,dz|$$ ◇

We now have an easy but extremely useful bound: the *ML–inequality*.

Lemma 2.36. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a contour. Suppose that $f\colon U \to \mathbb{C}$ is continuous. We take*

$$M = \max\{|f(\gamma(t))| : t \in [a,b]\}$$

Then we have:

$$\left|\int_\gamma f\,dz\right| \leq M \cdot L(\gamma)$$

Exercise 2.37. Prove Lemma 2.36. ◇

3. Variation of argument

> Customer: *I'd like to have an argument, please.*
> Receptionist: *Certainly sir. Have you been here before?*
> Customer: *No, this is my first time.*
> Receptionist: *I see. Do you want to have the full argument, or were you thinking of taking a course?*
>
> – Monty Python
> *Argument Clinic*

We have two ways to understand the variation of argument: analytical and topological.

3.1. **The analytical viewpoint.** Define the *element of argument* by:

$$d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

This can be derived by differentiating $\tan(\theta) = y/x$.

Definition 3.2. Suppose that $\gamma = (\gamma_x, \gamma_y)\colon [a,b] \to \mathbb{C}^\times$ is a contour. We define the *variation of argument* of γ, about zero, as follows:

$$V(\gamma) = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_a^b \frac{\gamma_x(t)\gamma_y'(t) - \gamma_y(t)\gamma_x'(t)}{|\gamma(t)|^2}\,dt \qquad \diamondsuit$$

For example, the variation of argument about zero for the upper half-circle (oriented anti-clockwise) is π; the variation for the whole circle is 2π.

Lemma 3.3. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour. Suppose that $\delta\colon [c,d] \to [a,b]$ is piecewise C^1. Suppose that $\gamma \circ \delta$ is again a contour. Suppose that $\delta(c) = a$ and $\delta(d) = b$. Then we have $V(\gamma \circ \delta) = V(\gamma)$. If, instead, $\delta(c) = b$ and $\delta(d) = a$ then $V(\gamma \circ \delta) = -V(\gamma)$.* $\square$

The proof is similar to that of Lemma 2.27 and we omit it. The next proposition is another step in the direction of finding holomorphic branches of the logarithm (Section **??**); this should be compared to Lemma 1.11.

Proposition 3.4. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a contour. Suppose that $\theta_a \in \textsc{Arg}(\gamma(a))$ is an argument. Then there exists a unique continuous $\theta\colon [a,b] \to \mathbb{R}$ so that*

- *$\theta(a) = \theta_a$ and*
- *$\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ for all $t \in [a,b]$.*

For this function θ, we have $V(\gamma) = \theta(b) - \theta(a)$.

Proof. Induction on the number of vertices reduces the general case to the case where there are no vertices in the interior of γ. That is, we may assume that γ is C^1. For s in $[a,b]$ we define $\gamma_s = \gamma|[a,s]$. Next we define

$$\theta(s) = \theta_a + V(\gamma_s) = \theta_a + \int_{\gamma_s} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

We deduce from the definition that $V(\gamma) = \theta(b) - \theta(a)$. We also deduce, from the real fundamental theorem of calculus, that θ is differentiable (as a real function of s). Thus θ is continuous.

We define $r(s) = |\gamma(s)|$. We now claim that $\gamma(s) = r(s)e^{i\theta(s)}$ for all $s \in [a,b]$. To prove this, consider

$$h(s) = \frac{r(s)e^{i\theta(s)}}{\gamma(s)}$$

Differentiating h we find that $h'(s) = 0$ for all s.

Exercise 3.5. Check this. $\diamondsuit$

Also $h(a) = 1$; thus $h(s) = 1$ for all s. This implies that $\gamma(s) = r(s)e^{i\theta(s)}$ for all s.

Suppose that $\eta\colon [a,b] \to \mathbb{R}$ is another such function. That is, $\eta(a) = \theta_a$ and $\gamma(t) = r(t)e^{i\eta(t)}$ for all $t \in [a,b]$. We deduce that $\theta(t) - \eta(t)$ is continuous and also lies in $2\pi\mathbb{Z}$ for all t. Thus $\theta(t) - \eta(t)$ is constant. However $\theta(a) - \eta(a) = 0$, so $\eta = \theta$, as desired. □

Lemma 3.6. *Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a closed contour. Then we have $V(\gamma) \in 2\pi\mathbb{Z}$.*

Proof. Recall that $\gamma(b) = \gamma(a)$. Pick any argument $\theta_a \in \textsc{Arg}(\gamma(a))$; apply Proposition 3.4 to build the function θ. We deduce that $e^{i\theta(b)} = e^{i\theta(a)}$. The addition law, and periodicity, tells us that $V(\gamma) = \theta(b) - \theta(a)$ lies in $2\pi\mathbb{Z}$, as desired. □

Definition 3.7. Suppose that $\gamma\colon [a,b] \to \mathbb{C}^\times$ is a closed contour. We define

$$\textsc{Wind}(\gamma) = V(\gamma)/2\pi$$

to be the *winding number* of γ about zero. ◇

It follows from Lemma 3.6 that the winding number is an integer.

Remark 3.8. Suppose that $z_0 \in \mathbb{C}$. Suppose that $\gamma\colon [a,b] \to \mathbb{C} - \{z_0\}$ is a contour missing z_0. Then we define $V(\gamma, z_0)$, the variation of argument of γ about z_0; it is the variation of argument of $t \mapsto \gamma(t) - z_0$ about zero.

Suppose that γ is closed (and, again, misses z_0). Then we define $\textsc{Wind}(\gamma, z_0)$, the winding number of γ about z_0; it is the winding number of $t \mapsto \gamma(t) - z_0$ about zero. ◇

3.9. **The topological viewpoint.**

Second proof of Proposition 3.4. Recall that the image of γ avoids the origin. For every $t \in [a,b]$ we choose some r_t so that (the closure of) the disc $B(\gamma(t); r_t)$ misses the origin. Let I_t be the component of $\gamma^{-1}(B(\gamma(t); r_t))$ that contains t. So $\mathcal{U} = \{I_t\}$ is an open cover of $[a,b]$ by intervals. Since $[a,b]$ is compact, we may choose a finite subcover $\mathcal{V} \subset \mathcal{U}$. Let $T \subset [a,b]$ be the finite set of times t so that I_t lies in $\mathcal{V}$.

We now define a continuous function $\theta\colon [a,b] \to \mathbb{R}$ by induction. We first define $\theta(a) = \theta_a$. Suppose now that we have defined the function $\theta|[a,c]$ on the interval $[a,c] \subset [a,b]$. Suppose additionally that $\theta|[a,c]$ is continuous and satisfies $\theta(s) \in \textsc{Arg}(\gamma(s))$ for all $s \in [a,c]$.

If $c = b$ then take $\theta = \theta|[a,b]$ and the definition is complete. Suppose instead that $c < b$. Now, c lies in (the interior of) I_t for some $t \in T$. Let c' be the supremum of I_t. Note that $\gamma([c,c'])$ lies in (the closure of) $B(\gamma(t); r_t)$. We apply Exercise 1.10 and the triangle inequality: for all s in $[c,c']$ there is a unique argument $\theta(s) \in \textsc{Arg}(\gamma(s))$ so that $|\theta(s) - \theta(c)| < \pi$. This extends $\theta|[a,c]$ to $[a,c']$. By Lemma 1.11 the extension $\theta|[a,c']$ is continuous. By construction $\theta(s)$ lies in $\textsc{Arg}(\gamma(s))$

for all $s \in [a, c']$. Finally, since T is finite this process terminates, giving the desired $\theta | [a, b]$.

The argument for the uniqueness of θ (subject to the condition that $\theta(a) = \theta_a$) is identical to one given above. □

3.10. **Element of the logarithm.** We end this section with three more examples.

The *element of radius* is:

$$dr = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

This can be derived by differentiating $r^2 = x^2 + y^2$. Suppose that $\gamma\colon [a, b] \to \mathbb{C}^\times$ is a contour in the punctured plane. Then integrating the element of radius over γ gives the difference $|\gamma(b)| - |\gamma(a)|$.

The *element of logarithmic radius* is:

$$\frac{dr}{r} = \frac{x\,dx + y\,dy}{x^2 + y^2}$$

This can be derived by differentiating $\log(r)$. Integrating this over γ, a contour in $\mathbb{C}^\times$, gives the logarithm of the ratio $|\gamma(b)|/|\gamma(a)|$.

Finally, the *element of the logarithm* is:

$$\frac{dr}{r} + i\,d\theta = \frac{dz}{z}$$

This differential has already appeared in the solution of Exercise 3.5.

4. Holomorphic functions

4.1. **Limits in $\mathbb{C}$.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that z_0 is a point of U. Suppose that $g\colon U - \{z_0\} \to \mathbb{C}$ is a function. Then we write

$$\lim_{z \to z_0} g(z) = c$$

if, for all real $\epsilon > 0$ there is some real $\delta > 0$ with the following property: if $0 < |z - z_0| < \delta$ then $|g(z) - c| < \epsilon$.

Note that it is *not* enough to consider sequences tending to z_0 along a particular line, or finite collection of lines. Instead, we must allow z to tend to z_0 inside of a shrinking sequence of (punctured) discs.

We now have the central definition of these notes.

Definition 4.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. We say that f is *holomorphic* if, for all $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. ◇

We denote the limit of the difference quotients, at z_0, by $f'(z_0)$. Since f is holomorphic in all of U, we obtain a function $f' \colon U \to \mathbb{C}$. We call f' the *complex derivative* of f. Note that we do *not* assume that f' is continuous.

Exercise 4.3. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Then f is continuous. ◇

The following observation is "trivial" but also very useful.

Lemma 4.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Fix $z_0 \in U$. Define $\rho \colon U \to \mathbb{C}$ by*

$$\rho(z) = \begin{cases} 0 & z = z_0 \\ \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) & z \neq z_0 \end{cases}$$

Then ρ is continuous.

Exercise 4.5. Prove Lemma 4.4. ◇

Remark 4.6. With Exercise 4.3 and Lemma 4.4 in hand, the very beginning of the theory of complex analysis exactly follows that of real analysis in one variable. In particular we have the following:

- constant functions are holomorphic;
- the identity function $f(z) = z$ is holomorphic;
- sums, products, and compositions of holomorphic functions are holomorphic and the derivatives obey the sum, product, and chain rules; and
- ratios of holomorphic functions are holomorphic (away from the zeros of the denominator) and the derivatives obey the quotient rule.

We deduce that polynomials $P, Q \in \mathbb{C}[z]$ are holomorphic, as is the rational function P/Q, away from the roots of Q. ◇

As a special case of the last point we have an exceedingly important example: the function $f(z) = 1/z$ is holomorphic in the punctured plane $\mathbb{C}^\times$.

Example 4.7. As a non-example, consider $f(z) = \bar{z}$. Its derivative at zero, taken along the real axis, is 1. Its derivative at zero, taken along the imaginary axis, is -1. Thus $f(z) = \bar{z}$ is not holomorphic. (It is sometimes called "anti-holomorphic".) ◇

Exercise 4.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that $\gamma \colon [a, b] \to U$ is a C^1 contour with no internal breakpoints. Prove that the derivative of $f \circ \gamma$, at t, is $f'(\gamma(t)) \cdot \gamma'(t)$. ◇

Exercise 4.9. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that $f'(z) = 0$ for all $z \in U$. Then f is constant. ◇

As an application of Lemma 4.4 we show that holomorphic functions are differentiable as real-valued functions of two variables.

Lemma 4.10. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then, for any $w \in U$, the total derivative $D_w f$ is as follows:*

$$D_w f = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$$

Here $r = |f'(w)|$ and, if $f'(w) \neq 0$, then θ is any argument of $f'(w)$.

Proof. Applying Lemma 4.4 at $z_0 = w$ we have $f(z) = f(w) + f'(w)(z - w) + \rho(z)(z - w)$. So the total derivative $D_w f$ (a two-by-two matrix with real entries) acts exactly as multiplication by $f'(w)$ does. □

So $D_w f$, when it is non-zero, is a homothety (scale-and-rotation). Thus f, away from the zeros of its derivative, is *conformal*: it preserves signed angles.

4.11. **Power series.** We quickly review the theory of power series.

Definition 4.12. Suppose that z_0 is a complex number. Suppose that $(a_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers. Then we write

$$\sum_{k\in\mathbb{N}} a_k (z - z_0)^k$$

for the *power series* with *centre* z_0 and *coefficients* (a_k). We write $f_n(z) = \sum_{k=0}^{n} a_k (z - z_0)^k$ for the n^{th} *partial sum*. We also define

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k}$$

We take $R = 0$ or $R = \infty$ when the limit superior is infinite or zero, respectively. We call R the *radius of convergence* of the given power series. ◇

Ahlfors [1, Theorem 2, page 38] attributes the following to Abel.

Theorem 4.13. *Suppose that z_0, $(a_k)_k$, f_n, and R are as in Definition 4.12. Suppose that $R > 0$. Suppose that $0 < R' < R$. Then:*

- *The power series converges absolutely for all z in $B(z_0; R)$.*

Let f be the resulting function on $B(z_0; R)$.

- *The partial sums f_n converge uniformly to f on $\overline{B(z_0; R')}$.*
- *The function f is holomorphic in $B(z_0; R)$.*
- *Its derivative $f'(z)$ has the power series*

$$\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

  which again has radius of convergence R.
- *The function f has complex derivatives of all orders at z_0; its k^{th} derivative is $f^{(k)}(z_0) = k! \cdot a_k$.* □

We omit the proofs; they are identical to those in the one-variable real case. See [2, Theorems 9.20, 21, 23; pages 234–237], or [12, Theorems 3.9, 10, 16, 19, 21; pages 44–52], or [5, Theorems 10.3, 6; pages 184–186].

If f is as in Theorem 4.13, then we say that f has a *series expansion* in $B(z_0; R)$. Note that the function f also has a series expansion in $B(z_0; R')$ for all $R' < R$. The algebra of functions and of power series "agree", as follows.

Theorem 4.14. *Suppose that z_0 is a complex number. Suppose that $R > 0$. Suppose that $f, g\colon B(z_0; R) \to \mathbb{C}$ are functions. Suppose that f and g have series expansions in $B(z_0; R)$ with coefficients $A = (a_k)_k$ and $B = (b_k)_k$, respectively. Then:*

- *f and g are equal if and only if A and B are equal.*
- *$h = f + g$ has a series expansion in $B(z_0; R)$ with coefficients $C = (a_k + b_k)_k$.*
- *$h = f \cdot g$ has a series expansion in $B(z_0; R)$ with coefficients $C = (c_n)_n$ where*

$$c_n = \sum_{k+\ell=n} a_k b_\ell$$

- *Suppose that m is a non-negative integer. Suppose that $f = (z - z_0)^m \cdot g$. Then:*

$$a_k = \begin{cases} 0, & \text{for } k < m \\ b_{k-m} & \text{for } k \geq m \end{cases}$$

- *Suppose that $f(z_0) \neq 0$. Suppose that $h = 1/f$. Then there is some $S > 0$ so that h has a series expansion in $B(z_0; S)$ with coefficients $C = (c_n)_n$ where*

$$a_0 c_0 = 1 \quad \text{and, for } n > 0 \quad \sum_{k+\ell=n} a_k c_\ell = 0 \qquad \square$$

We again omit the proofs; they are again identical to those in the one-variable real case. See [2, Theorems 9.24, 26; pages 237–240], or [12, Theorems 3.18, 55; pages 49–50, 71–72], or [5, Proposition 10.15, 16; pages 190–192].

Exercise 4.15. Give a power series about zero for the function $f(z) = 1/(1 + z^2)$. Determine its radius of convergence. $\diamondsuit$

Exercise 4.16. Give a power series about zero for the function $f(z) = 1/(1 + z + z^2)$. Determine its radius of convergence. $\diamondsuit$

We define the complex versions of the exponential and the trigonometric functions as follows.

Definition 4.17. We define

$$\textsc{Exp}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\textsc{Cos}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\textsc{Sin}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$ ◇

These agree with their real siblings when restricted to the real axis in $\mathbb{C}$. The family resemblances extend much further.

Lemma 4.18. *The complex exponential and trigonometric functions have the following properties. (Here z and w range over $\mathbb{C}$ unless otherwise stated.)*

- $\textsc{Cos}(0) = 1$ *and* $\textsc{Sin}(0) = 0$.
- $\textsc{Cos}(2\pi) = 1$ *and* $\textsc{Sin}(2\pi) = 0$.
- $\textsc{Cos}'(z) = -\textsc{Sin}(z)$ *and* $\textsc{Sin}'(z) = \textsc{Cos}(z)$.
- $\textsc{Cos}^2(z) + \textsc{Sin}^2(z) = 1$.
- $\textsc{Exp}(iz) = \textsc{Cos}(z) + i\,\textsc{Sin}(z)$.
- $\textsc{Exp}(0) = 1$ *and* $\textsc{Exp}(2\pi i) = 1$.
- $\textsc{Exp}'(z) = \textsc{Exp}(z)$.
- $\textsc{Exp}(z+w) = \textsc{Exp}(z)\,\textsc{Exp}(w)$.
- *The addition laws:*

$$\textsc{Cos}(z+w) = \textsc{Cos}(z)\,\textsc{Cos}(w) - \textsc{Sin}(z)\,\textsc{Sin}(w)$$
$$\textsc{Sin}(z+w) = \textsc{Cos}(z)\,\textsc{Sin}(w) + \textsc{Sin}(z)\,\textsc{Cos}(w)$$

- $\textsc{Cos}(z+2\pi) = \textsc{Cos}(z)$ *and* $\textsc{Sin}(z+2\pi) = \textsc{Sin}(z)$.
- *All periods of* Cos *and* Sin *are multiples of* 2π.
- $\textsc{Exp}(z+2\pi i) = \textsc{Exp}(z)$.
- *All periods of* Exp *are multiples of* $2\pi i$.
- $\textsc{Exp}(z) \neq 0$.
- *For all* $w \in \mathbb{C}^\times$ *there is some* $z \in \mathbb{C}$ *so that* $\textsc{Exp}(z) = w$.
- $\textsc{Exp}(-z) = 1/\textsc{Exp}(z)$.
- Exp *is a surjective homomorphism from the group* $(\mathbb{C}, +)$ *to the group* $(\mathbb{C}^\times, \cdot)$, *with kernel* $(2\pi i\mathbb{Z}, +)$. □

Exercise 4.19. Prove that the zeros of the complex cosine and sine are all real. (So they agree with the zeros of the real cosine and sine, respectively). ◇

Exercise 4.20. Prove that the complex exponential takes "rectangular coordinates" (in its domain) to "polar coordinates" (in its codomain). That is, it takes horizontal line segments to radial ones and vertical line segments to arcs of circles (centred on the origin).

Suppose that $0 < a < b$ are real numbers. Sketch the closed rectangle R with vertices at $\log(a)$, $\log(b)$, $\log(b) + 2\pi i$, and $\log(a) + 2\pi i$. Sketch the image of R and its boundary under the complex exponential. ◇

In view of the above, for $w \in \mathbb{C}^\times$ we define

$$\textsc{Log}(w) = \{z \in \mathbb{C} \mid \textsc{Exp}(z) = w\}$$

So the complex logarithm Log is a "multi-valued function", like the argument Arg.

Lemma 4.21. *Suppose that $w \in \mathbb{C}^\times$. Then we have:*

$$\textit{Log}(w) = \log(|w|) + i\,\textit{Arg}(w)$$ □

5. Triangles

> *Egli [l'universo] è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.*
>
> – Galileo Galilei
> *Il Saggiatore*

5.1. **Segments.** Suppose that p and q are complex numbers. We use the notation

$$[p, q] = \{(1 - t)p + tq \mid t \in [0, 1]\}$$

to denote the *line segment* between p and q. The segment is a set. However, we also treat it as a contour using the linear parametrisation $t \mapsto (1 - t)p + tq = p + t(q - p)$. Note that $[p, q]$ and $[q, p]$ are identical as sets but have opposite orientations as contours.

Example 5.2. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} dz = \int_0^1 (q - p)\,dt = q - p$$

This is a special case of Exercise 2.26. ◇

Example 5.3. Suppose that p and q are complex numbers. Then we have:

$$\int_{[p,q]} z\,dz = \int_0^1 (p+t(q-p))(q-p)\,dt = p(q-p)+\frac{1}{2}(q-p)^2 = \frac{1}{2}(q^2-p^2)$$

◇

In fact, if $\gamma\colon [a, b] \to \mathbb{C}$ is any contour, then

$$\int_\gamma z\,dz = \frac{1}{2}(\gamma(b)^2 - \gamma(a)^2)$$

This will follow from Lemma 7.8 once we have the concept of *primitives* in hand.

5.4. **Convexity.** Suppose that K is a subset of the complex plane. We say that K is *convex* if, for every $p, q \in K$, the line segment $[p, q]$ lies in K. As a first example, the plane itself is convex.

Exercise 5.5. Prove that both $B(z_0; r)$ and $\overline{B(z_0; r)}$ are convex. ◇

Suppose that S is a set of complex numbers. We define the *convex hull* of S to be the intersection of all convex sets containing S.

Exercise 5.6. Prove that the convex hull of S is convex. (This justifies the name. We also deduce that the convex hull of S is the smallest convex set containing S.) ◇

Definition 5.7. Suppose that p, q, and r are complex numbers. The *triangle* $T = T(p, q, r)$ is the convex hull of $\{p, q, r\}$. By Exercise 5.6 the line segments $[p, q]$, $[q, r]$, and $[r, p]$ lie in T; these are the *sides* of T. We call p, q, and r the *vertices* of T. We define the *polygonal boundary* ∂T to be the contour given by traversing the segments $[p, q]$, $[q, r]$, and $[r, p]$ in that order and with those orientations.

If p, q, and r lie in a line in $\mathbb{C}$ then we call T a *degenerate triangle.* ◇

Lemma 5.8. *Suppose that a, b, p, q, and r are complex numbers. Take $f(z) = a + bz$; take $T = T(p, q, r)$. Then we have:*

$$\int_{\partial T} f \, dz = 0$$

Proof. Lemma 2.22 implies the following.

$$\int_{\partial T} (a + bz) \, dz = a \int_{\partial T} dz + b \int_{\partial T} z \, dz$$

Lemma 2.25 implies the following.

$$\int_{\partial T} dz = \int_{[p,q]} dz + \int_{[q,r]} dz + \int_{[r,p]} dz$$
$$\int_{\partial T} z \, dz = \int_{[p,q]} z \, dz + \int_{[q,r]} z \, dz + \int_{[r,p]} z \, dz$$

Examples 5.2 and 5.3 give the following.

$$\int_{\partial T} dz = (q - p) + (r - q) + (p - r) = 0$$
$$\int_{\partial T} z \, dz = \frac{1}{2}[(q^2 - p^2) + (r^2 - q^2) + (p^2 - r^2)] = 0$$

Thus the integral of $f(z) = a + bz$ about ∂T is zero, as desired. □

Remark 5.9. Lemma 5.8 generalises in many ways. As one possibility, suppose that $f \in \mathbb{C}[z]$ is a polynomial. Suppose that $T \subset \mathbb{C}$ is a triangle. Then $\int_{\partial T} f \, dz = 0$.

As another possibility, fix $R > 0$. Suppose that $f \colon B(0; R) \to \mathbb{C}$ is given by a power series. Suppose that the triangle T lies in $B(0; R)$. Recall (from Theorem 4.13) that the partial sums f_n converge uniformly

to f on closed discs. Uniform convergence allows us to interchange the integral and the limit, giving $\int_{\partial T} f\,dz = 0$.

We can also generalise the contour from the boundary of a triangle to *any* closed contour in $B(0;R)$. All of these statements are eventually superseded by Theorem 14.11. ◇

5.10. **Orientation.**

> *And should not I spare Nineveh, that great city, wherein are more than sixscore thousand persons that cannot discern between their right hand and their left hand; and also much cattle?*
>
> – Jonah 4:11
> *King James Bible*

Suppose that u and v are complex numbers. We define their *cross-product* as follows:

$$u \times v = \textsc{Imag}(\bar{u}v)$$

If we identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way we obtain the usual cross-product. We use this to give a simple definition of "handedness"; for one reference among many, see [6, page 3].

Definition 5.11. We define $\mathcal{O}\colon \mathbb{C}^3 \to \mathbb{R}$ as follows:

$$\mathcal{O}(u,v,w) = u \times v + v \times w + w \times u$$

Suppose that (u,v,w) lies in $\mathbb{C}^3$. We say that the ordered triple (u,v,w) is an *anti-clockwise triple* if $\mathcal{O}(u,v,w)$ is positive. It is *clockwise* if $\mathcal{O}(u,v,w)$ is negative. Finally, it is *degenerate* if $\mathcal{O}(u,v,w)$ vanishes. ◇

For example, the three cube roots of unity, in the usual order

$$\left(1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

form an anti-clockwise triple.

Exercise 5.12. Suppose that a, b, u, v, and w are complex numbers. Prove the following:

- $\mathcal{O}(v,w,u) = \mathcal{O}(u,v,w)$
- $\mathcal{O}(v,u,w) = -\mathcal{O}(u,v,w)$
- $\mathcal{O}(u,v,w) = 0$ if and only if u, v, and w are co-linear.
- $\mathcal{O}(au+b, av+b, aw+b) = |a|^2\mathcal{O}(u,v,w)$ ◇

Definition 5.13. Suppose that p, q, and r are complex numbers. Suppose that $T = T(p,q,r)$ is a non-degenerate triangle. We say that T is *positively oriented* if (p,q,r) is an anti-clockwise triple. Otherwise T is *negatively oriented*. ◇

Remark 5.14. It is a pleasant exercise to show that $\mathcal{O}(p,q,r)$ is twice the (signed) area of the triangle $T(p,q,r)$. We do not need this fact, so we leave the proof to the interested reader. ◇

Exercise 5.15. Suppose that L is an oriented line in $\mathbb{C}$. Define what it means for a point $w \in \mathbb{C} - L$ to be *to the left of* L.

Suppose that $T = T(p, q, r)$ is a positively oriented triangle. Suppose that L is the oriented line in $\mathbb{C}$ containing $[p, q]$. Prove that all points of $T - L$ are to the left of L. $\diamond$

Suppose that $T = T(p, q, r)$ is a non-degenerate triangle and $z_0 \in T^\circ$ is any point of the interior. Remark 3.8, and a computation, imply that $\textsc{Wind}(\partial T, z_0) = \pm 1$ as T is positively or negatively oriented.

5.16. **Midpoint subdivision of a triangle.**

Definition 5.17. Suppose that $T = T(p, q, r)$ is a triangle. Let

$$p' = (q + r)/2 \qquad q' = (r + p)/2 \qquad r' = (p + q)/2$$

be the midpoints of the sides of T. We define four subtriangles of T as follows:

$$P = T(p, r', q') \qquad Q = T(r', q, p')$$
$$R = T(q', p', r) \qquad S = T(p', q', r')$$

The collection $\textsc{Mid}(T) = \{P, Q, R, S\}$ is the *midpoint subdivision* of the triangle T. $\diamond$

The triangles P, Q, R, and S are all similar to T (under scaling by one-half and, for S, a half-turn) and all have the same orientation as T. So all four smaller triangles are positive, negative, or degenerate as T is. For a labelled sketch, see Figure 6.4.

Suppose that $\gamma\colon [a, b] \to \mathbb{C}$ is a contour. Recall that $L(\gamma)$ is its arc-length (Definition 2.29). We use $D(\gamma)$ for the *diameter* of γ:

$$D(\gamma) = \max\{|\gamma(s) - \gamma(t)|\colon s, t \in [a, b]\}$$

Lemma 5.18. *Suppose that T is a triangle in $\mathbb{C}$. Suppose that P, Q, R, and S are the four triangles of the midpoint subdivision $\textsc{Mid}(T)$. Then we have*

$$\frac{1}{2}L(\partial T) = L(\partial P) = L(\partial Q) = L(\partial R) = L(\partial S)$$
$$\frac{1}{2}D(\partial T) = D(\partial P) = D(\partial Q) = D(\partial R) = D(\partial S) \qquad \square$$

Remark 5.19. Suppose that $L = [a, b]$ is a line segment. We define

$$\textsc{Mid}(L) = \{[a, (a + b)/2], [(a + b)/2, b]\}$$

Note that midpoint subdivision of a triangle "induces" the midpoint subdivision of its boundary segments. The internal segments produced by the subdivision have opposite orientations (in pairs). So these cancel against each other when we take contour integrals. Said another way: "taking boundaries commutes with midpoint subdivision". In symbols, we might write this as $\partial(\textsc{Mid}(T)) = \textsc{Mid}(\partial T)$.

One version of this is proved in Exercise 6.6 and another in Section 8.20. $\diamond$

6. Goursat's lemma

Voici une démonstration du théorème de Cauchy, qui me parait un peu plus simple que les démonstrations habituelles. Elle repose uniquement sur la définition de la dérivée et sur cette remarque que les intégrales définies $\int dz$, $\int z\,dz$, prises le long d'un contour fermé quelconque, sont nulles.

– Édouard Goursat
Démonstration du théorème de Cauchy

6.1. **Integrating to zero around triangles.**

Definition 6.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a continuous function. We say that f *integrates to zero around triangles* if, for every triangle $T \subset U$, we have

$$\int_{\partial T} f\,dz = 0 \qquad \diamondsuit$$

We state and prove a result of Pringsheim, correcting a result of Goursat, improving upon a result of Cauchy. For references, and an overview of the history, see [4].

Lemma 6.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then f integrates to zero around triangles in U.*

That is, "holomorphic functions integrate to zero around triangles".

Proof of Lemma 6.3. In the proof we will use the following notations. Suppose that $\gamma\colon [a,b] \to U$ is a contour. We define $I(\gamma) = \int_\gamma f\,dz$. As in Definition 2.29, we use $L(\gamma)$ for the arc-length of γ. We again use $D(\gamma)$ for the diameter of the image of γ.

By Exercise 4.3 we have that f is continuous. We now fix a triangle $T = T(p,q,r) \subset U$. We must prove that

$$I(\partial T) = \int_{\partial T} f\,dz = 0$$

Suppose that P, Q, R, and S are the triangles of $\textsc{Mid}(T)$, the midpoint subdivision of T (Definition 5.17). As the other cases are similar or easier, suppose that ∂T, and the boundaries of the smaller triangles, are oriented as in Figure 6.4.

Claim 6.5. $I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$

Proof. We apply Lemmas 2.25 and 2.27 and the combinatorics of Figure 6.4. □

Exercise 6.6. Give a more detailed proof of Claim 6.5. $\diamondsuit$

Taking absolute values and applying the triangle inequality, we have

$$|I(\partial T)| \leq |I(\partial P)| + |I(\partial Q)| + |I(\partial R)| + |I(\partial S)|$$

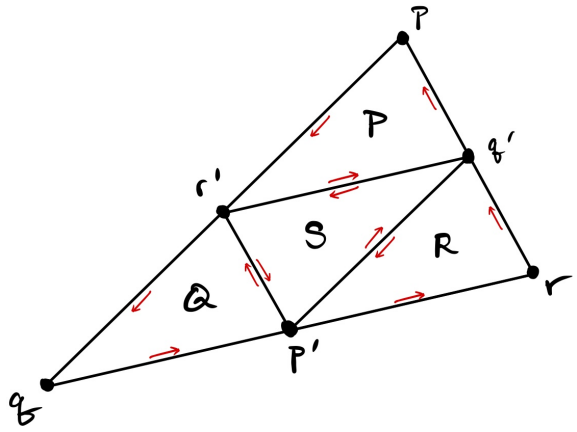

FIGURE 6.4. The midpoint subdivision of T. We label the vertices of all of the triangles. We also decorate the edges of the four smaller triangles with their orientations.

Set $T_0 = T$. Pick T_1 to be any one of P, Q, R, or S so that

$$|I(\partial T_1)| = \max\{|I(\partial P)|, |I(\partial Q)|, |I(\partial R)|, |I(\partial S)|\}$$

We deduce that

$$|I(\partial T_1)| \geq \frac{1}{4}|I(\partial T_0)|$$
$$L(\partial T_1) = \frac{1}{2}L(\partial T_0)$$
$$D(\partial T_1) = \frac{1}{2}D(\partial T_0)$$

We repeat the above process of subdivision indefinitely. This gives us a sequence of triangles $(T_n)_{n=0}^{\infty}$ so that, for all n, we have $T_{n+1} \subset T_n$. Also:

$$|I(\partial T_{n+1})| \geq \frac{1}{4}|I(\partial T_n)|$$
$$L(\partial T_{n+1}) = \frac{1}{2}L(\partial T_n)$$
$$D(\partial T_{n+1}) = \frac{1}{2}D(\partial T_n)$$

Exercise 6.7. The intersection of the T_n consists of a single point. ◇

Let z_0 be that point. Induction gives us, for all n:

$$|I(\partial T_n)| \geq \frac{1}{4^n}|I(\partial T_0)|$$
$$L(\partial T_n) = \frac{1}{2^n}L(\partial T_0)$$
$$D(\partial T_n) = \frac{1}{2^n}D(\partial T_0)$$

Recall that f is the given holomorphic function. We take $\rho\colon U \to \mathbb{C}$ as in Lemma 4.4. Thus ρ is continuous, vanishes at z_0, and satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)$$

Fix any $\epsilon > 0$. We deduce that there is a $\delta > 0$ so that $\overline{B(z_0;\delta)}$ is contained in U and for $w \in \overline{B(z_0;\delta)}$ we have $|\rho(w)| \leq \epsilon$. Since the triangles (T_n) nest down to z_0 there is some n so that T_n is contained in $\overline{B(z_0;\delta)}$. We now compute:

$$\begin{aligned}
I(\partial T_n) &= \int_{\partial T_n} f\,dz && \text{definition of } I(\partial T_n)\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)]\,dz && \text{Lemma 4.4}\\
&= \int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0)]\,dz + \int_{\partial T_n} \rho(z)(z - z_0)\,dz && \text{Lemma 2.22}\\
&= \int_{\partial T_n} \rho(z)(z - z_0)\,dz && \text{Lemma 5.8}
\end{aligned}$$

Now, the maximum of $|\rho|$ on ∂T_n is at most ϵ. The maximum of $|z - z_0|$ in ∂T_n is at most $D(\partial T_n)$. Thus, by the ML–inequality (Lemma 2.36) we have:

$$|I(\partial T_n)| \leq \epsilon \cdot D(\partial T_n) \cdot L(\partial T_n) = \frac{\epsilon}{4^n} \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Thus:

$$|I(\partial T_0)| \leq \epsilon \cdot D(\partial T_0) \cdot L(\partial T_0)$$

Since $\epsilon > 0$ was arbitrary, we deduce that $I(\partial T_0) = 0$, as desired. □

Exercise 6.8. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that R is a rectangle in U. Orient the boundary ∂R anti-clockwise. Prove that $\int_{\partial R} f\,dz = 0$. ◇

That is, "holomorphic functions integrate to zero around rectangles".

7. Primitives

Far out in the uncharted backwaters of the unfashionable end of the western spiral arm of the Galaxy lies a small unregarded yellow sun. Orbiting this at a distance of roughly ninety-two million miles is an utterly insignificant little blue green planet whose ape-descended life forms are so amazingly primitive that they still think digital watches are a pretty neat idea.

– Douglas Adams
The Hitchhiker's Guide to the Galaxy

7.1. **Piecewise-linear paths.** Suppose that $p, q \in \mathbb{C}$ are complex. Suppose that $a < b \in \mathbb{R}$ are real. We define the *affine* contour from p to q by

$$\gamma(t) = \frac{b-t}{b-a} \cdot p + \frac{t-a}{b-a} \cdot q$$

More generally, we say that a contour $\gamma\colon [a,b] \to \mathbb{C}$ is *piecewise linear* if there are finitely many breakpoints $(t_k)_{k=0}^N \subset [a,b]$ with $t_0 = a$, with $t_k < t_{k+1}$, with $t_N = b$, and with all pieces $\gamma|[t_k, t_{k+1}]$ affine.[3]

7.2. **Polygons.** Suppose that P is a non-empty, compact, connected subset of $\mathbb{C}$. Suppose that P°, the interior of P, is dense in P. Suppose finally that the topological boundary ∂P is the image of a simple, closed, piecewise-linear contour. Then we call P a *(simple) polygon.*

Suppose that γ_0 is a piece of ∂P (as in Definition 2.12). Suppose that z_0 is the midpoint of γ_0. Choose $\epsilon > 0$ small enough so that $B(z_0;\epsilon) \cap \partial P$ is a single diameter of $B(z_0;\epsilon)$ and is contained in γ_0. So one component of $B(z_0;\epsilon) - \gamma_0$ lies in P° and the other does not. (This requires the hypothesis that P° is dense in P and that z_0 lies in the boundary.) We orient γ_0 so that the component in P° is to the left of γ_0 (as in Exercise 5.15).

This induces an orientation on all of ∂P. We call this the *orientation of ∂P induced by γ_0*. Note that this is independent of the choice of ϵ (as long as it is sufficiently small).

We now decompose ∂P into a minimal possible collection of segments (β_k), starting with the one containing γ_0. We call the (β_k) the *sides* of P. We order these so β_{k+1} follows β_k in ∂P. We call the resulting contour the *polygonal boundary of P*. Finally we define p_k via $\{p_k\} = \beta_{k-1} \cap \beta_k$. We call the p_k the *vertices* of P.

As a special case, non-degenerate triangles are polygons. More generally, any convex hull of finitely many points (at least three, not all in a line) gives a polygon.

Exercise 7.3. Show that P can be obtained as a finite union of triangles $(T_\ell)_\ell$ so that:

- each T_ℓ is non-degenerate,
- distinct T_ℓ and $T_{\ell'}$ have disjoint interiors, and
- distinct T_ℓ and $T_{\ell'}$ are either disjoint, meet in a single vertex, or meet in a common side. ◇

Suppose that $(T_\ell)_\ell$ is such a decomposition of P. We orient all of the T_ℓ positively (Definition 5.13). This induces the anti-clockwise orientation of each contour ∂T_ℓ. So by Exercise 5.15 each triangle is to the left of its sides. We now "walk" around the boundary of P

- starting at any triangle meeting an interval β of ∂P containing z_0,

[3]This should be called "piecewise affine", but nobody says that.

- going from triangle to adjacent triangle, and
- always keeping ∂P immediately to our right.

By our convention, the side of the first triangle meeting β agrees with the orientation on β. Induction (and walking about vertices) implies that any side of any T_ℓ, that lies in some side of ∂P, receives the same orientation from both. Thus the orientation given to ∂P above did not depend on the initial choice of γ_0. We say that the orientation of ∂P is *induced* by P; it is also called the *positive orientation* and also the *anti-clockwise orientation*.

Equipped with these definitions, we can give a motivating special case of Theorem 9.2.

Exercise 7.4. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that $P \subset U$ is a polygon, with polygonal boundary ∂P oriented as given above. Prove that $\int_{\partial P} f\,dz = 0$. $\diamondsuit$

That is, "holomorphic functions integrate to zero around polygons".

7.5. **Basic properties of primitives.** Here is our next definition.

Definition 7.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that $F\colon U \to \mathbb{C}$ is holomorphic. Suppose that $F' = f$. Then we say that F is a *primitive* for f in U. $\diamondsuit$

Here we are following the terminology of [10, page 170] and [13, page 78]. Another name used for $f\,dz$ is "exact differential" [1, page 107].

Lemma 7.7. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that $F, G\colon U \to \mathbb{C}$ are both primitives for f. Then $F - G$ is constant.*

Proof. Note that F and G are holomorphic and thus so is $F - G$ (Remark 4.6). We have

$$(F - G)' = F' - G' = f - f = 0$$

So Exercise 4.9 implies that $F - G$ is constant. $\square$

We require the following version of the "fundamental theorem of calculus".

Lemma 7.8. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Suppose that $F\colon U \to \mathbb{C}$ is a primitive for f. Suppose that $\gamma\colon [a, b] \to U$ is a contour. Then*

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. We write $F(z) = u(z) + iv(z)$. Note that $u \circ \gamma$ and $v \circ \gamma$ are real C^1 functions, so obey the real fundamental theorem of calculus. Recall

that $F'(\gamma(t))\gamma'(t) = (F \circ \gamma)'(t)$, by Exercise 4.8. We now compute as follows:

$$\begin{aligned}
\int_\gamma f\,dz &= \int_\gamma F'\,dz \\
&= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_a^b (F \circ \gamma)'(t)\,dt \\
&= \int_a^b (u \circ \gamma + i\,v \circ \gamma)'(t)\,dt \\
&= \int_a^b [(u \circ \gamma)'(t) + i\,(v \circ \gamma)'(t)]\,dt \\
&= \int_a^b (u \circ \gamma)'(t)\,dt + i \int_a^b (v \circ \gamma)'(t)\,dt \\
&= [u \circ \gamma(b) - u \circ \gamma(a)] + i\,[v \circ \gamma(b) - v \circ \gamma(a)] \\
&= F(\gamma(b)) - F(\gamma(a)) \qquad \square
\end{aligned}$$

Lemma 7.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then f has a primitive in U if and only if $\int_\gamma f\,dz = 0$ for all closed contours γ in U.*

Proof. Suppose that f has a primitive F. Suppose that $\gamma\colon [a,b] \to U$ is a closed contour. Then $F(\gamma(b)) = F(\gamma(a))$ and we are done by Lemma 7.8.

Suppose instead that f integrates to zero about all closed contours in U. Fix w_0 in U. We now define a function $F\colon U \to \mathbb{C}$: for any $w \in U$ pick a piecewise-linear contour $\beta\colon [0,1] \to U$ with $\beta(0) = w_0$ and $\beta(1) = w$. (Note that β exists by Exercise 2.4.) We define $F(w) = \int_\beta f\,dz$. Suppose now that α is another such contour. Define γ to be the concatenation of β and the reverse of α:

$$\gamma(t) = \begin{cases} \beta(2t), & 0 \le t \le 1/2 \\ \alpha(2-2t), & 1/2 \le t \le 1 \end{cases}$$

So γ is a closed contour in U. Thus $\int_\gamma f\,dz = 0$. By Lemmas 2.25 and 2.27 we have

$$0 = \int_\gamma f\,dz = \int_\beta f\,dz - \int_\alpha f\,dz \quad \text{and thus} \quad \int_\beta f\,dz = \int_\alpha f\,dz$$

We deduce that F is well-defined.

Suppose that w_1 lies in U. We must prove that $F'(w_1) = f(w_1)$. Fix $\epsilon > 0$. Since f is continuous, and since U is open, we have a δ as follows: if $0 < |w - w_1| < \delta$ then $w \in U$ and $|f(w) - f(w_1)| \le \epsilon$.

Suppose that β is a piecewise-linear contour from w_0 to w_1. (Again, β exists by Exercise 2.4.) Let $[w_1, w]$ be the line segment from w_1 to w. Thus $[w_1, w]$ lies in U. Let β_w be the concatenation of β and $[w_1, w]$. So

$$F(w_1) = \int_\beta f\,dz \quad \text{and} \quad F(w) = \int_{\beta_w} f\,dz$$

Thus

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We subtract $f(w_1)(w - w_1)$ from both sides to find

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} f(z) - f(w_1)\,dz$$

The ML–inequality (Lemma 2.36) implies that

$$|F(w) - F(w_1) - f(w_1)(w - w_1)| \leq \epsilon |w - w_1|$$

Dividing both sides by $|w - w_1|$ gives

$$\left| \frac{F(w) - F(w_1)}{w - w_1} - f(w_1) \right| \leq \epsilon$$

Thus $F'(w_1) = f(w_1)$ and we are done. □

Exercise 7.10. Suppose that $f\colon \mathbb{C}^\times \to \mathbb{C}$ is given by $f(z) = 1/z$. Prove that f does not have a primitive in $\mathbb{C}^\times$. ◇

7.11. **Primitives in discs.**

Lemma 7.12. *Suppose that $B = B(w_0; r)$ is a disc. Suppose that $f\colon B \to \mathbb{C}$ is continuous. Suppose that f integrates to zero about all triangles in B. Then f has a primitive in B.*

Note that the disc B cannot be replaced by an arbitrary domain U. On one hand, by Lemma 6.3, the function $f(z) = 1/z$ integrates to zero about all triangles in $\mathbb{C}^\times$. On the other hand, by Exercise 7.10, we know that f has no primitive in $\mathbb{C}^\times$. On the third hand, Lemma 7.12 implies that f has a primitive in every disc that avoids the origin.

Proof of Lemma 7.12. Let $[w_0, w]$ be the line segment from w_0 to w. We define $F(w) = \int_{[w_0,w]} f\,dz$.

Suppose that w_1 is another point of B. We must prove that $F'(w_1) = f(w_1)$. Since f integrates to zero about triangles, we have:

$$F(w) - F(w_1) = \int_{[w_1,w]} f\,dz$$

We now subtract $f(w_1)(w - w_1)$ from both sides to obtain

$$F(w) - F(w_1) - f(w_1)(w - w_1) = \int_{[w_1,w]} (f(z) - f(w_1))\,dz$$

The rest of the proof is similar to that of Lemma 7.9. □

8. Homology

The following pages are intended to exhibit some of the advantages obtained by a more extensive use of topological methods and notions in courses on complex variables.

– Emil Artin
On the theory of complex functions

8.1. **Simplices and affine maps.** The *standard n–simplex* is

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x_k \geq 0, \sum x_k = 1 \right\}$$

The k^{th} *vertex* v_k of Δ^n is the vector with all coordinates equal to zero, except for the k^{th}, which is equal to one. Note that Δ^n has $n+1$ vertices.

Suppose that X is a topological space. A *singular n–simplex σ^n in X* is a continuous map $\sigma^n \colon \Delta^n \to X$. Suppose that W is a real vector space. A singular n–simplex σ^n in W is *linear* if σ^n is the restriction of a linear map.

Lemma 8.2. *Suppose that W is a real vector space. Suppose that (w_k) is an ordered collection of $n+1$ points in W. Then there is a unique linear n–simplex σ^n where $\sigma^n(v_k) = w_k$, for all k.* □

8.3. **Codimension-one faces.** Fix $k \leq n$. We define the list

$$L_k = [v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n]$$

The notation here means that v_k is *omitted* from the list. Thus L_k has n elements, in order.

Suppose that σ^n is a singular n–simplex in X. By Lemma 8.2 there is a unique linear $(n-1)$–simplex

$$\tau_k^{n-1} \colon \Delta^{n-1} \to \mathbb{R}^{n+1}$$

where τ_k^{n-1} takes the ℓ^{th} vertex of Δ^{n-1} to the ℓ^{th} vertex of L_k. The notation for, and the definition of, the k^{th} $(n-1)$–dimensional *face* of σ^n is:

$$\sigma^n|[v_0, v_1, v_2, \ldots, v_{k-1}, \hat{v}_k, v_{k+1}, \ldots, v_{n-1}, v_n] = \sigma^n \circ \tau_k^{n-1}$$

Remark 8.4. We will almost always suppress the notation for the reparametrising map τ_k^{n-1}. ◇

Most relevant to these notes are singular simplices in dimensions zero, one, and two.

- A singular zero-simplex σ^0 has no -1–dimensional faces.
- A singular one-simplex σ^1 has two zero-dimensional faces; these are $\sigma^1|[v_1]$ and $\sigma^1|[v_0]$.
- A singular two-simplex σ^2 has three one-dimensional faces; these are $\sigma^2|[v_1, v_2]$, $\sigma^2|[v_0, v_2]$, and $\sigma^2|[v_0, v_1]$.

Exercise 8.5. List the four two-dimensional faces of a singular three-simplex σ^3. ◇

8.6. **Chains and the boundary operator.**

Definition 8.7. Suppose that X is a topological space. Fix n. An *n–chain* in X is a formal, finite sum of singular n–simplices in X with integer coefficients. The collection of all n–chains in X is a commutative group denoted $C_n(X)$; the empty sum gives the additive identity (that is, zero). ◇

Definition 8.8. Suppose that X is a topological space. Fix n. Suppose that σ^n is a singular n–simplex in X. We define the *homological boundary* of σ^n as follows:

$$\partial_n \sigma^n = \sum_k (-1)^k \sigma^n | [v_0, v_1, \ldots, \hat{v}_k, \ldots, v_n]$$

So the homological boundary is the $(n-1)$–chain obtained by summing the $(n-1)$–faces of σ^n, with sign. We extend ∂_n linearly to n–chains to obtain a homomorphism

$$\partial_n \colon C_n(X) \to C_{n-1}(X)$$

We call this the n^{th} *boundary operator*. ◇

In the relevant dimensions (for us) we have:

$$\begin{aligned}
\partial_0 \sigma^0 &= 0 \\
\partial_1 \sigma^1 &= \sigma^1 | [v_1] - \sigma^1 | [v_0] \\
\partial_2 \sigma^2 &= \sigma^2 | [v_1, v_2] - \sigma^2 | [v_0, v_2] + \sigma^2 | [v_0, v_1]
\end{aligned}$$

It is a nice exercise to prove, for all $n \geq 1$, that $\partial_{n-1} \circ \partial_n = 0$. However, we will only need this fact in dimension one.

Exercise 8.9. Prove that $\partial_1 \circ \partial_2 \colon C_2(X) \to C_0(X)$ is the trivial homomorphism. ◇

8.10. **Cycles and boundaries.**

Definition 8.11. An n–chain $c \in C_n(X)$ is an *n–cycle* if $\partial_n c = 0$. An n–chain $c \in C_n(X)$ is an *n–boundary* if there is some $d \in C_{n+1}(X)$ so that $\partial_{n+1} d = c$. ◇

We define $Z_n(X) < C_n(X)$ to be the kernel of the homomorphism ∂_n; this is the subgroup of n–cycles. We define $B_n(X) < C_n(X)$ to be the image of the homomorphism ∂_{n+1}; this is the subgroup (again of $C_n(X)$) of n–boundaries. Furthermore, because $\partial_n \circ \partial_{n+1} = 0$, we have the following:

$$B_n(X) < Z_n(X) < C_n(X)$$

We may therefore define, as a quotient, the n^{th} *(singular) homology group* of X:

$$H_n(X) = Z_n(X)/B_n(X)$$

(This is one of the main ways of transforming problems in topology into problems in algebra.) The elements of $H_n(X)$ are called n–dimensional *homology classes* in X. We will only need these in dimension $n = 1$.

8.12. **Examples of cycles and boundaries.** When discussing the various groups (of chains, cycles, boundaries, and homology classes) one must *always* specify the ambient space X. We explore this in a collection of examples.

Example 8.13. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $\gamma\colon [a,b] \to U$ is a closed contour. Using Lemma 8.2, let $p\colon \Delta^1 \to [a,b]$ be the unique linear simplex sending v_0 to a and v_1 to b. Let $c_\gamma = \gamma \circ p$. Thus c_γ is a one-chain in U. Since γ is a closed contour we find that c_γ is a one-cycle in U. ◇

Remark 8.14. As another notational saving, similar in spirit to Remark 8.4, we will often suppress the (formally necessary) reparametrisation making Δ^1 the domain instead of an interval of $\mathbb{R}$.

We will, conversely, also allow ourselves to treat (piecewise C^1) one-simplices as contours; here we will also suppress the (formally necessary) reparametrisation by an interval of $\mathbb{R}$. ◇

Example 8.15. Suppose that $\iota\colon \mathbb{C}^\times \to \mathbb{C}$ is the inclusion mapping. Consider the contour $\gamma\colon \Delta^1 \to \mathbb{C}^\times$ given by $\gamma(x_0, x_1) = e^{2\pi i x_1}$. Note that $\gamma((1,0)) = \gamma((0,1))$; so γ is a one-cycle in $\mathbb{C}^\times$. We define $\delta = \iota \circ \gamma$; so δ is a one-cycle in $\mathbb{C}$.

We claim that δ is a one-boundary in $\mathbb{C}$. To see this, first define $c\colon \Delta^2 \to \mathbb{C}$ by

$$c(x_0, x_1, x_2) = \begin{cases} 0, & \text{if } x_2 = 1 \\ (1 - x_2)e^{2\pi i \frac{x_1}{1-x_2}}, & \text{if } x_2 \neq 1 \end{cases}$$

Note that c is continuous away from the vertex $v_2 = (0,0,1)$. To prove that c is continuous at v_2, consider the arc $I_r \subset \Delta^2$ with $x_2 = 1 - r$. So $c|I_r$ parametrises $C(0;r)$, the circle of radius r about the origin. As r tends to zero (and so I_r tends to v_2) the circle $C(0;r)$ tends to the origin. Thus small neighbourhoods of v_2 are sent to small neighbourhoods of $c(v_2)$, as desired.

Note that $c|[v_0, v_1] = \delta$ (as singular one-simplices). Also, $c|[v_0, v_2] = c|[v_1, v_2]$ (also as singular one-simplices). We deduce that $\delta = \partial c$. Thus δ is a one-boundary in $\mathbb{C}$, as desired.

On the other hand, γ is not a one-boundary in $\mathbb{C}^\times$. The proof will be asked for in Exercise 9.3 after we have Theorem 9.2 in hand. ◇

Exercise 8.16. Check the claimed equality $c|[v_0, v_2] = c|[v_1, v_2]$ in Example 8.15. ◇

Exercise 8.17. Suppose that $\sigma^1 \colon \Delta^1 \to X$ is a singular one-simplex. Define $\tau^1(x_0, x_1) = \sigma^1(x_1, x_0)$. This is the *reverse* of σ^1. Prove that $\sigma^1 + \tau^1$ is a one-boundary. ◇

Example 8.18. Suppose that U is a domain. Suppose that $T = T(p, q, r)$ is a triangle in U. Inspired by Lemma 8.2 we define $\sigma^2_T \colon \Delta^2 \to U$ to be the linear two-simplex sending v_0 to p, v_1 to q, and v_2 to r. The homological boundary of σ^2_T is:

$$\partial\sigma^2_T = [p, q] + [q, r] - [p, r]$$

This is almost a realisation of the polygonal boundary of T. However, the last term is the negative of the reverse of what we actually want.

By Exercise 8.17 there is a two-chain $d \in C_2(U)$ so that $\partial d = [p, r] + [r, p]$. So

$$\partial(\sigma^2_T + d) = [p, q] + [q, r] + [r, p]$$

Thus the direct realisation of ∂T as a one-cycle is in fact a one-boundary. ◇

Exercise 8.19. Suppose that $\sigma^1 \colon \Delta^1 \to X$ is a singular one-simplex. Suppose that v is any point of Δ^1. Prove that $\sigma^1|[v_0, v] + \sigma^1|[v, v_1] - \sigma^1$ is a one-boundary. ◇

8.20. **Midpoint subdivision of a singular two-simplex.** Recall that Δ^2 is the standard triangle (in the plane $x_0 + x_1 + x_2 = 1$ in $\mathbb{R}^3$) with vertices v_0, v_1, and v_2. Let

$$v'_i = (v_j + v_k)/2 \qquad \text{for } \{i, j, k\} = \{0, 1, 2\}$$

be the midpoints of the sides of Δ^2. We define four subtriangles of Δ^2 as follows:

$$P = T(v_0, v'_2, v'_1) \qquad Q = T(v'_2, v_1, v'_0)$$
$$R = T(v'_1, v'_0, v_2) \qquad S = T(v'_2, v'_1, v'_0)$$

(Compare Definition 5.17 and note the change in the definition of S.) Identifying Δ^2 with any positive triangle in $\mathbb{C}$ we find that P, Q, and R have the same handedness as Δ^2 while S has the *opposite* handedness.

Suppose that U is a domain. Suppose that $\sigma^2 \colon \Delta^2 \to U$ is a singular two-simplex. We define the *midpoint subdivision* of σ^2 as follows:

$$\textsc{Mid}_2(\sigma^2) = \sigma^2|P + \sigma^2|Q + \sigma^2|R - \sigma^2|S$$

(The minus sign on $\sigma^2|S$ is due to the reversal of its handedness. Compare Example 8.18.) So $\textsc{Mid}_2(\sigma^2)$ is a two-chain (as discussed in Remark 8.4 we suppress the reparametrisations by Δ^2). We extend $\textsc{Mid}_2$ linearly to obtain a homomorphism $C_2(U) \to C_2(U)$. We similarly

define $\textsc{Mid}_1 \colon C_1(U) \to C_1(U)$, using the midpoint $v' = (v_0 + v_1)/2$ to subdivide Δ^1:

$$\textsc{Mid}_1(\sigma^1) = \sigma^1|[v_0, v'] + \sigma^1|[v', v_1]$$

(Again we suppress the reparametrisations by Δ^1.)

We can now explain the choice of handedness of S and sign of $\sigma^2|S$ in the definition of $\textsc{Mid}_2$.

Exercise 8.21. Prove that $\partial_2 \circ \textsc{Mid}_2 = \textsc{Mid}_1 \circ \partial_2$. $\diamond$

That is, taking boundaries commutes with the midpoint subdivision, even at the level of chains.

9. Cauchy's theorem

We adopt the following notation. Suppose that U is a domain. Suppose that $f \colon U \to \mathbb{C}$ is continuous. Suppose that $\Gamma = \sum g_k \sigma_k^1$ is a one-chain in U where all σ_k^1 are piecewise C^1. Then we write

$$\int_\Gamma f \, dz = \sum_k g_k \int_{\sigma_k^1} f \, dz$$

Using this we can state and prove one of the main results of the module.

Proposition 9.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is continuous. Suppose that f integrates to zero along all closed contours in all discs in U. Suppose that Γ is a one-boundary in U which is piecewise C^1. Then we have:*

$$\int_\Gamma f \, dz = 0$$

Proof. Since Γ is a one-boundary in U, following Definition 8.11, there is some two-chain $d = \sum_k d_k \sigma_k^2 \in C_2(U)$ so that $\Gamma = \partial d$. Recall that each σ_k^2 is continuous and Δ^2, the standard two-simplex, is compact. So $D_k = \sigma_k^2(\Delta^2)$ is compact and contained in U. By Exercise 2.6 there is a constant $\epsilon_k > 0$ so that the open neighbourhood $N(D_k; \epsilon_k)$ lies in U.

We define $\epsilon = \min_k\{\epsilon_k\}$. For each k there is some δ_k so that if $u, v \in \Delta^2$ have $|u - v| < \delta_k$ then $|\sigma_k^2(u) - \sigma_k^2(v)| < \epsilon$. We define $\delta = \min_k\{\delta_k\}$.

Now we subdivide. Let $T^{(N)} = \textsc{Mid}_2^N(\Delta^2)$ be the N^{th} midpoint subdivision of Δ^2. Every triangle of $T^{(N)}$ has diameter $\sqrt{2}/2^N$. Fix N so that $\sqrt{2}/2^N < \delta$.

Suppose that t is a triangle of $T^{(N)}$. We define $\sigma_{k,t}^2 = \sigma_k^2|t$ (as usual we suppress the notation arising from the need to reparametrise making Δ^2 the domain). So $\sigma_{k,t}^2$ is itself a singular two-simplex. Let $e_t = \pm 1$ be the corresponding sign (coming from Section 8.20). We define $(\sigma_k^2)^{(N)} = \sum_t e_t \sigma_{k,t}^2$ and we define $d^{(N)} = \sum_k d_k (\sigma_k^2)^{(N)}$.

We similarly define $\Gamma^{(N)} = \textsc{Mid}_1^N(\Gamma)$ to be the N^{th} midpoint subdivision of Γ. We now apply Exercise 8.21: taking boundaries commutes

with midpoint subdivision. So $\Gamma^{(N)} = (\partial d)^{(N)} = \partial(d^{(N)})$. By Lemmas 2.25 and 2.27 we have

$$\int_\Gamma f\,dz = \int_{\Gamma^{(N)}} f\,dz$$

Now we straighten. For any $\sigma^n\colon \Delta^n \to \mathbb{C}$ we define $\textsc{Str}_n(\sigma^n)\colon \Delta^n \to \mathbb{C}$ to be the real-linear map agreeing with σ^n at the vertices of Δ^n. We call $\textsc{Str}_n(\sigma^n)$ the *straightening* of σ^n. We extend linearly to obtain homomorphisms $\textsc{Str}_n\colon C_n(\mathbb{C}) \to C_n(\mathbb{C})$.

To lighten the notation a bit, we define $\tau_{k,t} = \textsc{Str}_2(\sigma^2_{k,t})$. So we have $\textsc{Str}_2(d^{(N)}) = \sum_k d_k \sum_t e_t \tau_{k,t}$. We also straighten $\Gamma^{(N)}$, obtaining $\textsc{Str}_1(\Gamma^{(N)})$. We set $d' = \textsc{Str}_2(d^{(N)})$ and $\Gamma' = \textsc{Str}_1(\Gamma^{(N)})$.

Straightening commutes with taking boundaries so we have:

$$\Gamma' = \textsc{Str}_1(\Gamma^{(N)}) = \textsc{Str}_1(\partial_2 d^{(N)}) = \partial_2\,\textsc{Str}_2(d^{(N)}) = \partial_2 d'$$

We must check that the new, straightened, simplices lie in U. Recall that t has diameter less than δ in Δ^2. Since $\tau_{k,t}$ is the straightening of $\sigma^2_{k,t}$ they agree at the vertices (of Δ^2). Suppose that v is the image of a vertex under either map. We deduce that the image of $\sigma^2_{k,t}$, and thus of $\tau_{k,t}$, lies in $B(v;\epsilon)$. Thus the image of $\tau_{k,t}$ lies in U. We deduce that $d' \in C_2(U)$ and so $\Gamma' \in B_1(U)$.

By Lemmas 2.25 and 2.27 the integral of f over $\partial\tau_{k,t}$ (the homological boundary) equals the integral of f over the polygonal boundary of (the image of) $\tau_{k,t}$. (Here we use Definition 5.7 and take vertices $p = \tau_{k,t}(v_0)$, $q = \tau_{k,t}(v_1)$, and $r = \tau_{k,t}(v_2)$.) But the latter is an integral over a closed contour in a disc. So it vanishes. That is,

$$\int_{\partial\tau_{k,t}} f\,dz = 0$$

Summing over k and t, we have

$$\int_{\Gamma'} f\,dz = 0$$

Suppose that η is a summand of $\Gamma^{(N)}$. Set $\eta' = \textsc{Str}_1(\eta)$. We deduce the following:

- η' is a linear one-simplex while η is a piecewise C^1 one-simplex,
- the endpoints and orientations of η' and η agree, and
- the images of both are contained in a single ϵ–disc in U.

The last follows for η because we have subdivided finely enough and follows for η' because it is a straightening. By Lemmas 2.25 and 2.27 the integral of f over $\eta' - \eta$ equals the integral of f over the concatenation of η' with the reverse of η. The concatenation is a closed contour in a disc, so the latter vanishes by our hypothesis on f. So we have

$$\int_{\eta'-\eta} f\,dz = 0$$

We deduce that

$$\int_{\eta'} f\,dz = \int_{\eta} f\,dz$$

Summing over summands of $\Gamma^{(N)}$, and recalling that Γ' is the straightening of $\Gamma^{(N)}$, we have

$$\int_{\Gamma'} f\,dz = \int_{\Gamma^{(N)}} f\,dz$$

This completes the proof. □

As a corollary we can give the more commonly-seen statement.

Theorem 9.2. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that Γ is a piecewise C^1 one-boundary in U. Then we have:*

$$\int_{\Gamma} f\,dz = 0$$

Proof. Suppose that f is holomorphic. By Exercise 4.3 we have that f is continuous. By Lemma 6.3 we have that f integrates to zero around triangles in discs in U. By Lemma 7.12 we have that f has primitives in discs in U. By Lemma 7.9 we have that f integrates to zero along all closed contours in all discs in U. We now apply Proposition 9.1 and are done. □

Here is our first application of Theorem 9.2; we prove that the choice of ambient space is crucial for deciding if a one-cycle is a one-boundary.

Exercise 9.3. With the notation as in Example 8.15: prove that γ is not a one-boundary in $\mathbb{C}^\times$. ◇

10. The first homology group

10.1. **Trivial first homology.** Suppose that $U \subset \mathbb{C}$ is a domain. We say that U has *trivial first homology* if $H_1(U) \cong 0$. That is, the first homology group of U is isomorphic to the trivial group. Digging into the definitions a bit, this happens if and only if every one-cycle in U is a one-boundary in U.

Here are some domains with trivial first homology.

Lemma 10.2. *Suppose that $U \subset \mathbb{C}$ is a convex domain. Then U has trivial first homology.*

Proof. Suppose that $c = \sum c_k \sigma_k^1$ is a one-cycle in U. We must find some $d \in C_2(U)$ so that $\partial d = c$.

Fix a point w in U. Suppose that $\sigma^1 \in C_1(U)$ is a singular one-simplex. We define $d_w(\sigma^1) \in C_2(U)$ following Example 8.15. That is:

$$d_w(\sigma^1)(x_0, x_1, x_2) = \begin{cases} w, & \text{if } x_2 = 1 \\ x_2 w + (1 - x_2)\sigma^1\left(\frac{x_0}{1-x_2}, \frac{x_1}{1-x_2}\right) & \text{if } x_2 \neq 1 \end{cases}$$

We define $d = \sum c_k d_w(\sigma_k^1)$. Note that this is a two-chain in U. If $\sigma_k^1|[v_1] = \sigma_j^1|[v_0]$ as zero-simplices, then $d_w(\sigma_k^1)|[v_1, v_2] = d_w(\sigma_j^1)|[v_0, v_2]$ as one-simplices. Also, $d_w(\sigma_k^1)|[v_0, v_1] = \sigma_k^1$, again as one-simplices. Since $\partial c = 0$ we deduce that $\partial d = \sum c_k \partial d_w(\sigma_k^1) = \sum c_k \sigma_k^1 = c$. So c is a one-boundary, as desired. □

As a consequence of the lemma, we have that any disc $B(z_0; r) \subset \mathbb{C}$ has trivial first homology. Similarly, the plane $\mathbb{C}$ has trivial first homology.

10.3. **Invariance.** Homology is an invariant of homeomorphism, as follows.

Exercise 10.4. Suppose that U and V are domains. Suppose that $f\colon U \to V$ and $g\colon V \to U$ are continuous and also inverses (thus both homeomorphisms). For any singular simplex $\sigma\colon \Delta^n \to U$ we define $f_*(\sigma) = f \circ \sigma$. We extend linearly to obtain $f_*\colon C_n(U) \to C_n(V)$.

Prove that f_* and g_* induce (inverse) isomorphisms between $H_1(U)$ and $H_1(V)$. ◇

As a special case, suppose that U is homeomorphic to V. Then $H_1(U) \cong 0$ if and only if $H_1(V) \cong 0$.

10.5. **Non-trivial first homology.** We turn to the task of giving a domain U so that $H_1(U)$ is non-trivial.

Definition 10.6. Suppose that $0 < a < b$ is a pair of real numbers. Suppose that z_0 is some complex number. We define

$$A(z_0; a, b) = \{z \in \mathbb{C} \mid a < |z - z_0| < b\}$$

This is the *(open) annulus* centred on z_0, with inner radius a, and with outer radius b. ◇

Example 10.7. In Exercise 9.3 we found a one-cycle in $\mathbb{C}^\times$ which is not a one-boundary in $\mathbb{C}^\times$. Thus $H_1(\mathbb{C}^\times) \ncong 0$. The same argument proves that annuli, and also punctured discs, have non-trivial first homology. This can also be deduced from Exercise 10.4. ◇

In fact, $H_1(\mathbb{C}^\times) \cong \mathbb{Z}$; the isomorphism is given by the winding number. Furthermore, domains with "more holes" have more homology. We do not pursue these lines of inquiry.

11. Regions

In presentations of Cauchy's theorem, the integral formula and the residue theorem which follow more classical lines, there is no mention of homology, nor is the notion of index used explicitly. Instead, the curve γ to which the theorems are applied is supposed to form the complete boundary of a subregion of Ω, and the orientation is chosen so that the subregion lies to the left of $[\gamma]$. In rigorous texts considerable effort is spent on proving that these intuitive notions have a precise meaning. The main objection to this procedure is the necessity to allot time and attention to rather delicate questions which are peripheral in comparison with the main issues.

– Lars V. Ahlfors
Complex analysis: An introduction to the theory of analytic functions of one complex variable

We generalise from closed discs, closed annuli, triangles, and polygons to the following.

Definition 11.1. Suppose that $R \subset \mathbb{C}$ satisfies the following:

- R is compact, connected, and non-empty;
- the interior, R°, is dense in R;
- the topological boundary, ∂R, is a finite union of disjoint simple closed contours; and
- each contour in ∂R has non-vanishing derivatives.

Then we call R a *region*. ◇

11.2. **Accessibility.** We show that the topological boundary ∂R of a region R is "accessible" from the interior, and from the exterior, of R. For a related result, see [14, Lemma 2.10, page 362].

Lemma 11.3. *Suppose that U is a domain. Suppose that $R \subset U$ is a region. Suppose that z_0 is a point of ∂R. Suppose that γ is the component of ∂R containing z_0. Then there is an $r_1 > 0$ so that, for all $0 < r < r_1$ we have the following:*

(1) the closure of $B = B(z_0; r)$ lies in U,
(2) the circle $C(z_0; r)$ meets ∂R in exactly two points,
(3) the two points of intersection lie in γ, and
(4) the difference $B - \gamma$ has two components: one lies in R and one is disjoint from R.

This statement should be compared with the (simpler!) situation of polygons and the induced orientation of their boundary (Section 7.2).

Proof of Lemma 11.3. We are given a parametrisation $\gamma\colon [a, b] \to \mathbb{C}$ with $\gamma'(t)$ non-vanishing. Suppose that t_0 lies in $[a, b]$ and that $\gamma(t_0) = z_0$. Reparametrising the closed contour (if needed), we may assume that $t_0 \neq a$ and $t_0 \neq b$. Fix any $0 < \eta_+ < b - t_0$ so that $(t_0, t_0 + \eta_+]$ contains no breakpoints of γ.

Set $\beta = \gamma|[t_0, t_0 + \eta_+]$. So β is C^1. Also, $\beta(t_0) = z_0$ and $\beta'(t_0) \neq 0$. Set $v = \beta'(t_0)$. Define $\rho\colon [t_0, t_0 + \eta_+] \to \mathbb{C}$ by

$$\rho(t) = \beta(t) - (z_0 + v(t - t_0))$$

So ρ is C^1. Also, $\rho(t_0) = 0$ and $\rho'(t_0) = 0$. Since ρ' is continuous, we have some $\delta_+ \in (0, \eta_+]$ with $|\rho'(t)| < |v|/6$ for all $t \in [t_0, t_0 + \delta_+]$. The (real) mean value theorem then implies that $|\rho(t)| \leq |v|(t - t_0)/6$ for all $t \in [t_0, t_0 + \delta_+]$.

We now consider the square of the distance:

$$\begin{aligned} S(t) &= |\beta(t) - z_0|^2 \\ &= |v(t - t_0) + \rho(t)|^2 \\ &= (v(t - t_0) + \rho(t))\Big(\overline{v}(t - t_0) + \overline{\rho(t)}\Big) \\ &= |v|^2(t - t_0)^2 + (t - t_0)\Big(\overline{v}\rho(t) + v\overline{\rho(t)}\Big) + |\rho(t)|^2 \end{aligned}$$

Note that this is real, as is its derivative. We differentiate to obtain

$$\begin{aligned} S'(t) = 2|v|^2(t - t_0) + (t - t_0)\Big(\overline{v}\rho'(t) + v\overline{\rho'(t)}\Big) + \Big(\overline{v}\rho(t) + v\overline{\rho(t)}\Big) \\ + \rho(t)\overline{\rho'(t)} + \rho'(t)\overline{\rho(t)} \end{aligned}$$

Subtracting $2|v|^2(t - t_0)$ from both sides and applying the triangle inequality we have:

$$\begin{aligned} &\Big|S'(t) - 2|v|^2(t - t_0)\Big| \\ &\qquad \leq |v|^2(t - t_0)\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{36} + \frac{1}{36}\right) \\ &\qquad \leq |v|^2(t - t_0) \end{aligned}$$

We deduce that, for all $t \in [t_0, t_0 + \delta_+]$, we have $S'(t) \geq |v|^2(t - t_0)$. So the square of the distance has positive derivative (away from $t = t_0$) and so is monotonically increasing. Thus the distance $|\beta(t) - z_0|$ is also monotonically increasing. Thus the contour β meets any circle $C(z_0; r)$ in one point, for any $r \leq |\beta(t_0 + \delta_+) - z_0|$.

We carry out the same argument on the left side of t_0, obtaining $\delta_- > 0$. Set

$$r_0 = \min\{|\gamma(t_0 - \delta_-) - z_0|, |\gamma(t_0 + \delta_+) - z_0|\}$$

Note that $\partial R - \gamma$ is closed, as is γ restricted to the complement of $(t_0 - \delta_-, t_0 + \delta_+)$, as is $\mathbb{C} - U$. So we may pick $0 < r_1 < r_0$ so that $\overline{B(z_0; r_1)}$ is disjoint from all three of these sets. So r_1 gives the first three conclusions of the lemma.

Suppose now that $0 < r < r_1$. Let $u(r)$ and $v(r)$ be the two points of $\gamma \cap C(z_0; r)$. We assume that these points come in the order $(u(r), z_0, v(r))$ along γ. By the monotonicity proved above, the

points $u(r)$ and $v(r)$ are continuous functions of r. We now define $w(r) \in C(z_0; r) - \{u(r), v(r)\}$ as follows:

- $w(r) \in z_0 + \textsc{Sqrt}((u(r) - z_0)(v(r) - z_0))$ and
- $(u(r), w(r), v(r))$ is an anti-clockwise triple.

Claim. $w(r)$ is well-defined.

Proof. Since $r > 0$ we have that $\mathbf{u} = u(r) - z_0$ and $\mathbf{v} = v(r) - z_0$ are non-zero. So the product $\mathbf{uv}$ has two square roots. Suppose that these are $\mathbf{p}$ and $\mathbf{q}$. Note that all of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{q}$ lie on the circle $C(0; r)$. Suppose that $\mathbf{v}/\mathbf{u}$ has argument 2θ. So there is a single rotation taking

$$\mathbf{u} \mapsto re^{-i\theta}, \quad \mathbf{v} \mapsto re^{i\theta}, \quad \mathbf{p} \mapsto \pm r, \quad \mathbf{q} \mapsto \mp r$$

Note that

$$\begin{aligned} \mathcal{O}(re^{-i\theta}, \quad r, re^{i\theta}) = \quad & 2r^2(1 - \cos(\theta)) \sin(\theta) \\ \mathcal{O}(re^{-i\theta}, -r, re^{i\theta}) = & -2r^2(1 + \cos(\theta)) \sin(\theta) \end{aligned}$$

Thus, rotating back, and appealing to Exercise 5.12, we find that $\mathcal{O}(\mathbf{u}, \mathbf{p}, \mathbf{v})$ and $\mathcal{O}(\mathbf{u}, \mathbf{q}, \mathbf{v})$ have opposite signs. Finally, translating back (by adding z_0) and again appealing to Exercise 5.12, we find that exactly one of the elements of

$$z_0 + \textsc{Sqrt}((u(r) - z_0)(v(r) - z_0))$$

forms an anti-clockwise triple with $u(r)$ and $v(r)$. □

We deduce that $(u(r), -w(r) + 2z_0, v(r))$ is a clockwise triple. We set $w(0) = z_0$.

Claim. The resulting function $w \colon [0, r_1) \to \mathbb{C}$ is continuous.

Proof. Since $w(r)$ lies in $C(z_0; r)$, we have that $w(r)$ tends to z_0 as r tends to zero. Thus w is continuous at zero.

Consider the curve $\alpha \colon (0, r_1) \to \mathbb{C}$ defined by $\alpha(r) = (u(r) - z_0)(v(r) - z_0)$. This is continuous and avoids the origin. Fix now any positive $r_2 < r_1$. Set $B_2 = B(\alpha(r_2); |\alpha(r_2)|/2)$. Applying Exercise 2.10, let $f \colon B_2 \to \mathbb{C}$ be the branch of $\textsc{Sqrt}$ so that $w(r_2) = z_0 + f(\alpha(r_2))$. Since u, v, f, α, and $\mathcal{O}$ are all continuous, we have some $\epsilon > 0$ so that $(r_2 - \epsilon, r_2 + \epsilon) \subset (0, r_1)$ and

- $\alpha(r) \in B_2$ and
- $\mathcal{O}(u(r), z_0 + f(\alpha(r)), v(r)) > 0$

for all $r \in (r_2 - \epsilon, r_2 + \epsilon)$. Thus $z_0 + f(\alpha(r)) = w(r)$ for all such $r \in (r_2 - \epsilon, r_2 + \epsilon)$. We deduce that w is continuous at r_2, as desired. □

Set $W_+ = w(0, r_1)$. By construction, W_+ is disjoint from ∂R. So, by the proof of the intermediate value theorem, W_+ is either contained in the interior of, or is disjoint from, the region R. The same applies to the arc $W_- = -w(0, r_1) + 2z_0$ (corresponding to the clockwise choices of square root).

Let $C_+(r)$ be the component of $C(z_0;r) - \{u(r), v(r)\}$ containing $w(r)$. Let $C_-(r)$ be the other component of $C(z_0;r) - \{u(r), v(r)\}$. Again the intermediate value theorem tells us that each circle-arc $C_\pm(r)$ is contained in the interior of, or is disjoint from, R.

Suppose that both W_+ and W_- are contained in R. Transitivity implies that all of the arcs $C_\pm(r)$ are also contained in R. So the points of $\partial R \cap B(z_0;r_1)$ are not accumulated by points of $\mathbb{C} - R$, a contradiction.

Suppose that both W_+ and W_- are disjoint from R. Transitivity implies that all of the circle-arcs $C_\pm(r)$ are also disjoint from R. So the points of $\partial R \cap B(z_0;r_1)$ are not accumulated by points of the interior of R, a contradiction.

We deduce that exactly one of $W_\pm$ is contained in R and the other is disjoint from R. The same holds, by transitivity, for the corresponding families of circle-arcs $C_\pm(r)$. This gives the final conclusion of the lemma. □

Remark 11.4. Lemma 11.3 is the one place where we use the assumption (in Definition 11.1) that the derivative of a contour appearing in the boundary of a region R is non-vanishing. With even more work, say using the implicit function theorem, we could show that ∂R has a *collar neighbourhood* inside of R. See, for example, [14, Theorem 2.9, page 361]. That would prove that regions are *surfaces with boundary*. ◇

11.5. **Induced orientations.** In Definition 5.13 and in Section 7.2 we gave the polygonal boundaries of triangles and polygons a preferred positive orientation, namely anti-clockwise. Regions can have many boundary components; so some care is required when defining their positive orientations.

Definition 11.6. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $R \subset U$ is a region. Suppose that γ is a component of ∂R. Suppose that z_0 is any point of γ. Suppose that B is a disc about z_0 as in Lemma 11.3. Suppose that u and v are the two points of $\gamma \cap \partial B$. Suppose that w is any point of $R^\circ \cap \partial B$. Orient γ so that it visits u before z_0 before v. Finally, if (u, v, w) is an anti-clockwise triple then we say that the orientation of γ is *positive*. ◇

We can say this more informally as follows: γ is oriented positively if R is locally to the left of γ at z_0.

Lemma 11.7. *The positive orientation of γ is well-defined.*

Proof. Suppose that z_0 and w_0 are two points of γ. Equip γ with the orientation making it positive at z_0. Suppose that δ is the closure of the component of $\gamma - \{z_0, w_0\}$ after z_0 (and so before w_0). Note that δ is compact. This and Lemma 11.3 gives finitely many points

$$(z_0, z_1, \ldots, z_N = w_0)$$

in order along δ and discs $B_k = \overline{B(z_k; r_k)}$ so that

- the intervals $\beta_k = B_k \cap \gamma$ cover δ,
- $\beta_k \cap \beta_\ell$ is non-empty if and only if $|k - \ell| \leq 1$,
- β_k neither contains nor is contained in β_{k+1}, and
- all triple intersections of discs are empty.

So suppose that the orientation of γ is positive at z_k. It suffices to prove that it is also positive at z_{k+1}.

Suppose that $C_k = \partial B_k$ and $C_{k+1} = \partial B_{k+1}$ are the two boundaries. From our assumptions on the (β_k) we deduce that C_k and C_{k+1} meet exactly twice.

From the hypotheses we have that γ meets, in order:

- some point of $B_k - B_{k+1}$, then
- the centre z_k, then
- some point of $B_k \cap B_{k+1}$, then
- the centre z_{k+1}, and finally
- some point of $B_{k+1} - B_k$.

We deduce that γ crosses the following arcs in the following order:

- $C_k - B_{k+1}$,
- $C_{k+1} \cap B_k$,
- $C_k \cap B_{k+1}$, and
- $C_{k+1} - B_k$.

See any one of the subfigures of Figure 11.8.

Lemma 11.3 implies that exactly one of the two points of $C_k \cap C_{k+1}$ lies in R. Let u and v be the points of $\gamma \cap C_k$, in order along γ. Let w be any point of $(C_k \cap R) - (\gamma \cup B_{k+1})$. See the first subfigure of Figure 11.8. The induction hypothesis tells us that (u, v, w) is an anti-clockwise triple. We now "shuffle" the points u, v, and w through the points of $C_k \cap C_{k+1}$ as shown in Figure 11.8. The first, second, fifth, and sixth moves happen on a single circle. In the third move the point v does not cross the line through u and w; in the fourth move the point w does not cross the line determined by u and v. So none of the moves change handedness. Thus after the moves (w, u, v) is still anti-clockwise, and we are done. □

Suppose that R is a region. The orientation on ∂R which is positive on all components of ∂R is called the *orientation induced by* R.

From the bird's-eye view, the induced orientation is anti-clockwise on the "outer" boundary of R and clockwise on all "inner" boundaries. The frog's-eye view is a bit different. Here we instead stand on γ (with our head just above the plane). We turn (if needed) so that R is on our *left*. This done, we are facing in the direction of the orientation on γ induced by R.

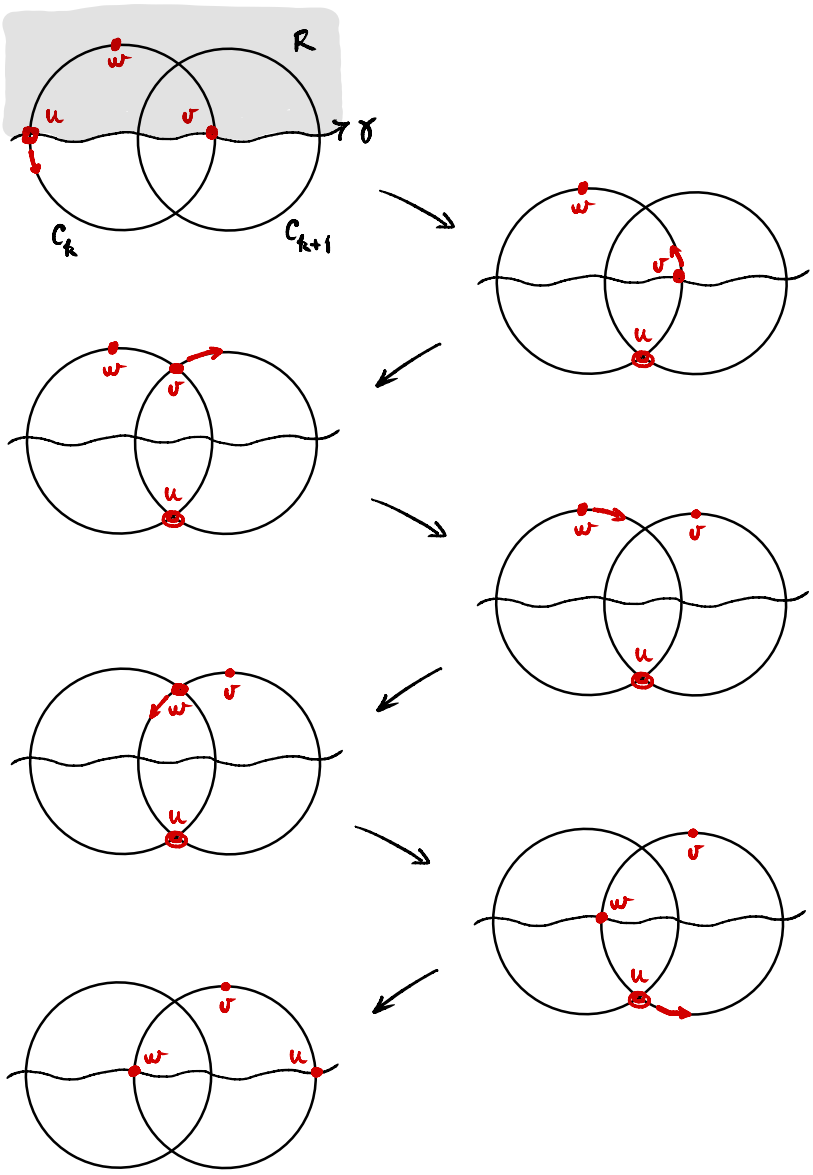

FIGURE 11.8. Shuffling points from one circle to the next.

Exercise 11.9. Go outside. Find a region R on the ground. Pick a component γ of ∂R. Run around γ in the direction of the orientation induced by R. ◇

12. CAUCHY'S THEOREM FOR REGIONS

To use Proposition 9.1 we need the following.

Proposition 12.1. *Suppose that U is a domain. Suppose that $R \subset U$ is a region; we equip ∂R with the orientations induced by R. Then ∂R, considered as a one-chain, is a one-boundary in U.*

There are several possible proofs of this. The conceptually simplest is to *triangulate* R: realise R as union of triangles, all nicely embedded in $\mathbb{C}$, all positively oriented, and all meeting nicely along their boundaries. This realises R as a two-chain in U and realises ∂R as a one-boundary in U. Triangulation can be achieved with techniques similar to those needed in building a collar (as in Remark 11.4). A version of this was given in last year's notes, but various subtle points were handled badly.

Another proof of Proposition 12.1 runs as follows. We first prove that, for any point, there is a nearby horizontal line L meeting ∂R in finitely many points. (This requires the C^1 hypothesis on ∂R.) Cut R along L into finitely many subregions. We repeat this process of cutting horizontally, and also vertically, until every subregion R_i is very small. At this stage each R_i is contained in some disc $B_i \subset U$. The components of ∂R_i are now closed contours in discs. Lemma 10.2 tells us that each ∂R_i is a one-boundary. We now sum these (and apply Exercises 8.17 and 8.19) to obtain the result.

Here is our third approach.

Proof of Proposition 12.1. Suppose that U is the given domain and $R \subset U$ is the given region. We equip ∂R with the orientations induced by R (Definition 11.6). For every point $z_0 \in \partial R$ we have a disc provided by Lemma 11.3. We shrink their radii by at least a factor of two to ensure that discs touching distinct components of ∂R are disjoint. We shrink even further to ensure that no component of ∂R is covered by only three such discs.

Note that the discs give an open cover of ∂R. So compactness provides us with a finite subcover (B_k). Take z_k to be the centre of B_k. Let $C_k = \partial B_k$.

Similar to the proof of Lemma 11.7, we may assume that

- no one of the B_k is contained in any other,
- every B_k meets exactly two others, and
- every component of ∂R meets at least four discs.

For each contour γ in ∂R we use the discs to form a new contour γ', as follows: if the discs B_k and B_ℓ meet each other and meet γ, then we place $[z_k, z_\ell]$ into γ'.

Claim 12.2. Each γ' is simple.

Proof. Fix one of the γ' built just above. We locally relabel the indices to get distinct discs $(B_n)_{n=0}^{N-1}$ covering γ'. These have centres (z_n) and radii (r_n). We treat the indices cyclically (ordered by γ) so that $\gamma' = \cup_n [z_n, z_{n+1}]$. By construction we have $N \geq 4$.

Suppose, for a contradiction, that γ' is not simple. Suppose that we have distinct indices m, $m+1$, n, and $n+1$ so that $[z_m, z_{m+1}]$ meets $[z_n, z_{n+1}]$. Let p be any point of the intersection. Since p lies on both

intervals we have

$$|z_m - p| + |p - z_{m+1}| = |z_m - z_{m+1}| < r_m + r_{m+1}$$
$$|z_n - p| + |p - z_{n+1}| = |z_n - z_{n+1}| < r_n + r_{n+1}$$

On the other hand, the distinctness of the four indices implies that $B_m \cap B_n = B_{m+1} \cap B_{n+1} = \varnothing$. So

$$|z_m - z_n| \geq r_m + r_n$$
$$|z_{m+1} - z_{n+1}| \geq r_{m+1} + r_{n+1}$$

Finally, the triangle inequality tells us

$$|z_m - z_n| = |z_m - p + p - z_n| \leq |z_m - p| + |p - z_n|$$
$$|z_{m+1} - z_{n+1}| = |z_{m+1} - p + p - z_{n+1}| \leq |z_{m+1} - p| + |p - z_{n+1}|$$

Chaining together the inequalities (and noting that one is strict) gives the desired contradiction.[4]

If some of the indices m, $m+1$, n, and $n+1$ agree, then (breaking symmetry) we may assume that we have $m + 1 = n$. Since γ' is not simple we deduce (as the other case is similar) that $[z_m, z_{m+1}]$ contains $[z_{m+1}, z_{m+2}]$. Recalling that $B_m \cap B_{m+2} = \varnothing$, and that no discs are nested, gives the desired contradiction. □

We orient γ' so that the centres z_k are in the same order on both γ and γ'.

Claim. The one-chain $\gamma - \gamma'$ is a one-boundary.

Proof. For any k, define $L_k = B_{k-1} \cap B_k$. Note that L_k is convex. Note also that γ meets L_k in a single sub-contour. Now pick any points $p_k \in \gamma \cap L_k$ and $p'_k \in [z_{k-1}, z_k] \cap L_k$. Let γ_k be the arc of γ from p_k to p_{k+1}. Let γ'_k be the arc of γ' from p'_k to p'_{k+1}. Let $[p_k, p'_k]$ be the line segment with those endpoints. We define the one-cycle

$$\zeta_k = \gamma_k + [p_{k+1}, p'_{k+1}] - \gamma'_k - [p_k, p'_k]$$

The image of ζ_k lies in B_k, so by Lemma 10.2 we have that ζ_k is a one-boundary. So $\sum \zeta_k$ is also a one-boundary. Noting the cancellation between ζ_k and ζ_{k+1} we find that

$$\sum \gamma_k - \sum \gamma'_k = \sum \zeta_k$$

is a one-boundary. Since (γ_k) and (γ'_k) are subdivisions of γ and γ' we find that

$$\gamma - \sum \gamma_k \qquad \text{and} \qquad \gamma' - \sum \gamma'_k$$

are both one-boundaries (Exercise 8.19). Thus $\gamma - \gamma'$ is a one-boundary, as desired. □

[4] In fact, for any convex quadrilateral, the sum of its diagonals is greater than the sum of a pair of opposite sides [9, Exercise 10, page 50].

We now build a region R' having the new contours as its topological boundary. We subtract from R the set $B_k \cap R$ and add back in the appropriate component of $B_k - \gamma'$.

We decompose R' into positively oriented triangles (T_ℓ) as in the solution to Exercise 7.3. We realise all of the triangles as linear two-simplices σ_ℓ^2. So $d' = \sum \sigma_\ell^2$ is a two-chain.

There are various ways the equality $\partial d' = \partial R'$ could fail. We consider these in turn.

Fix any index ℓ. Fix any summand e of $\partial\sigma_\ell^2$. There are two cases as the image of e meets the interior of R' or it does not.

Suppose first that e meets the interior of R'. So there is a unique index $\ell' \neq \ell$ and a unique summand e' in $\partial\sigma_{\ell'}^2$ so that e and e' have the same image. If e and e' have the same orientation and opposite sign then they cancel in the sum $\partial d'$ and we do not change d'. Otherwise, e and e' have opposite orientation and the same sign. In this subcase we apply Exercise 8.17 to find a two-chain d_e so that $\partial d_e = e + e'$. We replace d' by $d' \pm d_e$. Here we use the sign opposite to that of e, to ensure cancellation. (Also, we do not later consider e', to avoid undoing what we just did.)

Suppose next that e does not meet the interior of R'. So it lies in $\partial R'$. If its coefficient is $+1$ then its orientation agrees with that of $\partial R'$ and we do not change d'. If its coefficient is -1 then its orientation disagrees with that of $\partial R'$. Here we again form d_e and replace d' by $d' + d_e$.

Once we have so dealt with all indices ℓ and all summands e, we have that $\partial d'$ is a one-chain where all (non-cancelled) summands have

- positive sign and
- orientation agreeing with the induced orientation on $\partial R'$.

So $\partial d'$ gives a subdivision of $\partial R'$. We now apply Exercise 8.19 as needed, adding singular two-simplices to d', until $\partial d' = \partial R'$. Thus ∂R is a one-boundary, as desired. □

We are now equipped to prove the following.

Theorem 12.3. *Suppose that U is a domain. Suppose that $R \subset U$ is a region; we equip ∂R with the orientations induced by R. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Then $\int_{\partial R} f\,dz = 0$.*

That is, "holomorphic functions integrate to zero around regions".

Proof of Theorem 12.3. Exactly as in the proof of Theorem 9.2 we convert the hypothesis that f is holomorphic into the hypothesis that f integrates to zero along all closed contours in all discs in U. Propositions Proposition 9.1 and Proposition 12.1 complete the proof. □

13. The reproduction formula

We now use Theorem 12.3 to prove the *reproduction formula*. (This is also called the "Cauchy integral formula".)

Theorem 13.1. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $r > 0$ is a radius so that $\overline{B(z_0;r)}$ is contained in U. Let $C = C(z_0;r)$ be the boundary of $\overline{B(z_0;r)}$. Suppose that w lies in $B(z_0;r)$. Then we have:*

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

That is, the values that a holomorphic function f takes inside of a disc can be reproduced from the restriction of f to the boundary of the disc.

Proof of Theorem 13.1. Pick any $\delta_0 > 0$ so that $|z_0 - w| < r - \delta_0$. Pick any $0 < \delta < \delta_0$. So the circle $C(\delta) = C(w;\delta)$ is contained in $B(z_0;r)$. In particular, $C(\delta)$ is disjoint from C. We orient both C and $C(\delta)$ anti-clockwise.

By Exercise 2.5 we have that $U^\times = U - \{w\}$ is a domain. Take $A(\delta) = \overline{B(z_0;r)} - B(w;\delta)$. So $A(\delta)$ is closed, is homeomorphic to an annulus, and has $\partial A(\delta) = C \sqcup C(\delta)$. Both C and $C(\delta)$ are circles, so are both simple contours with non-vanishing derivative. Thus $A(\delta)$ is a region (Definition 11.1) in the domain $U^\times$. By Definition 11.6 and Lemma 11.7 the orientations induced by $A(\delta)$ on C and on $C(\delta)$ are anti-clockwise and clockwise, respectively. Note that $f(z)/(z-w)$ is holomorphic in $U^\times$. By Theorem 12.3 we have that

$$\frac{1}{2\pi i}\int_{\partial A(\delta)} \frac{f(z)}{z-w}\,dz = 0$$

Thus

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_{C(\delta)} \frac{f(z)}{z-w}\,dz$$

Let I_δ be the latter integral. We deduce that I_δ is independent of δ. On the other hand, Exercise 2.24 tells us that I_δ tends to $f(w)$ as δ tends to zero. Thus I_δ equals $f(w)$ (for all $\delta < \delta_0$) and we are done. □

13.2. **The mean value theorem.** Here is a useful special case of Theorem 13.1.

Theorem 13.3. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that z_0 is a point of U. Suppose that $r > 0$ is a radius so that $\overline{B(z_0;r)}$ is contained in U. Then we have:*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

In words: the value of a holomorphic function at a point is the average of its values on any circle about that point.

Proof of Theorem 13.3. In Theorem 13.1 we take $w = z_0$ and we parametrise $C = C(z_0; r)$ by $\gamma(\theta) = z_0 + re^{i\theta}$. We compute as follows:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(\theta))}{\gamma(\theta) - z_0} \gamma'(\theta)\, d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \end{aligned}$$ □

14. Analytic functions

14.1. **From holomorphic to analytic.**

Definition 14.2. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is a function. Suppose that for all $z_0 \in U$ there is some $r > 0$ so that

- $B(z_0; r) \subset U$ and
- f agrees with some power series in $B(z_0; r)$.

Then we call f *analytic*. ◇

From Theorem 4.13 we deduce the following.

Corollary 14.3. *Analytic functions are holomorphic.* □

Our goal in this section is to prove the converse.

Theorem 14.4. *Holomorphic functions are analytic.*

Proof. Here are our standing hypotheses and notations. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Fix a point z_0 in U. Fix r so that the closed disc $\overline{B(z_0; r)}$ lies in U. Let $C = C(z_0; r)$ be the boundary of $\overline{B(z_0; r)}$.

We define, for $n \in \mathbb{N}$:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

(By Theorem 13.1 we have $a_0 = f(z_0)$.) Set

$$M = \max\Big\{ |f(w)| : w \in \overline{B(z_0; r)} \Big\}$$

Lemma 14.5. $|a_n| \leq M/r^n$

Exercise 14.6. Prove Lemma 14.5. ◇

We deduce that the power series $F(w) = \sum_{n=0}^{\infty} a_n (w - z_0)^n$ converges in the disc $B(z_0; r)$.

Lemma 14.7. *For all $w \in B(z_0; r)$ we have $f(w) = F(w)$.*

Proof. We compute as follows:

$$\begin{aligned}
F(w) &= \sum_{n=0}^{\infty} a_n (w - z_0)^n && \text{definition } F(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right] (w - z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z - z_0)^{n+1}} (w - z_0)^n \right] dz && \text{interchange} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\sum \left(\frac{w - z_0}{z - z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \left[\frac{1}{1 - \frac{w - z_0}{z - z_0}} \right] dz && \text{geometric series} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - w} \, dz && \text{simplify} \\
&= f(w) && \text{Theorem 13.1}
\end{aligned}$$

The uniform convergence of the power series justifies the interchange of sum and integral. The geometric series $\sum \left(\frac{w - z_0}{z - z_0} \right)^n$ converges because w lies in the interior of $\overline{B(z_0; r)}$ while z lies on C, the boundary of $\overline{B(z_0; r)}$. □

This completes the proof of Theorem 14.4. □

We call $F(w)$, the power series appearing in the proof, the *series expansion of f about z_0*. Theorem 4.14 proves that F is unique, justifying the name.

Exercise 14.8. Give a direct proof that the interchange of sum and integral, in the proof of Theorem 14.4, is permitted. ◇

The proof of Theorem 14.4 gives an integral expression for the coefficients a_n of the series expansion. This, and Lemma 14.5, gives the following.

Corollary 14.9. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f \colon U \to \mathbb{C}$ is holomorphic. Suppose that $z_0 \in U$. Let R be the radius of convergence of the series expansion of f about z_0. Suppose that $B(z_0; r)$ is a subset of U. Then $r \leq R$.* □

In particular, if $U = \mathbb{C}$ then the radius of convergence is infinite. The converse does not quite hold.

14.10. **The fundamental theorem of complex analysis.** We gather our results so far into an "omnibus", inspired by [12, Theorem 1.1, page 3]. See [10, Theorem 8.2.1, page 236] for further equivalent properties.

Theorem 14.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is continuous. Then the following are equivalent.*

- *f is holomorphic in U*
- *f integrates to zero about triangles in U*
- *f has a primitive in every disc in U*
- *f integrates to zero about closed contours in discs in U*
- *f integrates to zero about (piecewise C^1) one-boundaries in U*
- *f integrates to zero about regions in U*
- *for all $z_0 \in U$ and $r > 0$ with $\overline{B(z_0; r)} \subset U$, for all $w \in B(z_0; r)$, and taking $C = C(z_0; r)$, we have*

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w}\, dz$$

- *f is analytic in U and, at a point $z_0 \in U$, the n^{th} coefficient of its series expansion is*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

where $C = C(z_0; \epsilon)$ lies in U. □

15. Zeros

15.1. **Orders of zeros.** Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic.

We have two pieces of "global" terminology.

Definition 15.2. Suppose that $f(z) = 0$ for all $z \in U$. Then we say that f *vanishes identically* in U.

On the other hand, suppose that $f(z) \neq 0$ for at least one $z \in U$. Then we say that f is not *identically zero* in U. ◇

We also have two bits of "local" terminology.

Definition 15.3. Suppose that z_0 is a point of U. Suppose that $A = (a_n)_n$ are the coefficients of the series expansion of f about z_0, as provided by Theorem 14.11.

Suppose that all of the a_n are zero. Then we say that f *vanishes to infinite order* at z_0.

On the other hand, suppose that not all of the a_n are zero. So there is a least N so that $a_N \neq 0$. We call N the *order of vanishing* of f at z_0; we denote this natural number N by $\textsc{Ord}(f, z_0)$. If the order is positive, then we call z_0 a *zero* of f of *order* $\textsc{Ord}(f, z_0)$. If the order is one, then we call z_0 a *simple zero* of f. ◇

A point $z_0 \in U$ where $\textsc{Ord}(f, z_0) = 0$ is sometimes called an *ordinary* or a *regular* point of f.

Lemma 15.4. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f vanishes to infinite order at $z_0 \in U$. Then f vanishes identically in U.*

Proof. Suppose that z_1 is another point of U.

Exercise 15.5. Prove that there is a constant $\epsilon > 0$ and a sequence of points $(w_k)_{k=0}^N \subset U$ as follows.

- $w_0 = z_0$,
- $w_N = z_1$,
- for all k we have $|w_{k+1} - w_k| < \epsilon$, and
- for all k the distance from w_k to any point of $\mathbb{C} - U$ is at least ϵ. ◇

With the points $(w_k)_{k=0}^N$ in hand, we induct. Applying Theorem 14.11 and Theorem 4.14, we take A_k to be the coefficients of the series expansion of f at w_k. By hypothesis, A_0 is the zero sequence. Suppose that, by induction, we have that A_k is also the zero sequence. Since the constant term is zero, we in particular have that $f(w_k) = 0$. If $k = N$ then $w_k = w_N = z_1$ and so $f(z_1) = 0$; in this case we are done. Suppose that $k < N$. Since A_k is the zero sequence, Theorem 4.13 tells us that f vanishes identically in $B(w_k; \epsilon)$. Note that w_{k+1} lies in $B(w_k; \epsilon)$. Thus all derivatives of f vanish at w_{k+1}. Applying Theorem 4.13 again, we find that A_{k+1} is also the zero sequence. This completes the induction step and the proof. □

15.6. **Patching.**

Lemma 15.7. *Suppose that U and V are domains in $\mathbb{C}$ with non-empty intersection. Then $U \cup V$ is again a domain in $\mathbb{C}$.* □

We can now state the "patching" lemma.

Lemma 15.8. *Suppose that U and V are domains with non-empty intersection. Suppose that $f\colon U \to \mathbb{C}$ and $g\colon V \to \mathbb{C}$ are holomorphic. Suppose further that $f|(U \cap V) = g|(U \cap V)$. Then there is a unique holomorphic function $h\colon U \cup V \to \mathbb{C}$ so that $h|U = f$ and $h|V = g$.*

Proof. We define h piecewise by

$$h(z) = \begin{cases} f(z), & z \in U \\ g(z), & z \in V \end{cases}$$

Our hypothesis implies that h is well-defined.

Suppose that z_0 is a point of $U \cup V$. Thus z_0 lies in U or in V. As the other case is similar, suppose that z_0 lies in U. Thus there is some $\epsilon > 0$ so that $B = B(z_0; \epsilon)$ is a subset of U. Since $h|B = f|B$ we find

that $h'(z_0) = f'(z_0)$. Thus h has a complex derivative at every point of $U \cup V$. So h is holomorphic on $U \cup V$. □

Remark 15.9. The conclusion of the proof relies on the fact that holomorphicity is a *local property*: that is, depending only on properties knowable from behaviour in arbitrarily small neighbourhoods of points. ◇

The patching lemma (15.8) is a holomorphic version of the *gluing lemma* in point-set topology: the latter builds continuous functions from continuous pieces.

15.10. **Factoring.**

Lemma 15.11. *Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic. Suppose that f is not identically zero in U. Fix $z_0 \in U$. Then there is a holomorphic function $g\colon U \to \mathbb{C}$ so that $g(z_0) \neq 0$ and so that*

$$f(z) = (z - z_0)^{\textsc{Ord}(f,z_0)} g(z)$$

Proof. Applying Lemma 15.4 we find that $\textsc{Ord}(f, z_0)$ is finite; suppose that f vanishes to order $\textsc{Ord}(f, z_0) = N > 0$ at z_0. Recall that f is holomorphic in U. So we have some $R > 0$ with $\overline{B(z_0; R)} \subset U$ and some coefficients $(a_n)_n$ giving a series expansion of f in $B(z_0; R)$. Thus $a_0 = a_1 = \ldots = a_{N-1} = 0$ while $a_N \neq 0$. Let $B = (b_n)_n$ be given by $b_n = a_{n+N}$. Appealing to Theorem 4.13 we define $g_0\colon B(z_0; R) \to \mathbb{C}$ to be the holomorphic function with power series with coefficients B. By Theorem 4.14 we have $f(z) = (z - z_0)^N g_0(z)$ on $B(z_0; R)$.

We next define $g_1\colon U - \{z_0\} \to \mathbb{C}$ by $g_1(z) = (z - z_0)^{-N} f(z)$. This is holomorphic by the quotient rule. Note that g_0 and g_1 agree on the punctured disc $B(z_0; R) - \{z_0\}$. Applying Lemma 15.8 gives the desired holomorphic $g\colon U \to \mathbb{C}$. □

Exercise 15.12. Suppose that $U \subset \mathbb{C}$ is a domain. Suppose that $g, f\colon U \to \mathbb{C}$ are holomorphic. Suppose that z_0 is a point of U. Prove that $\textsc{Ord}(fg, z_0) = \textsc{Ord}(f, z_0) + \textsc{Ord}(g, z_0)$ (where we allow either or both of f and g to vanish to infinite order at z_0). ◇

Appendix A. Solutions to exercises

Solution 1.4. Recall that $\cos\colon \mathbb{R} \to \mathbb{R}$ is continuous; also $\cos(0) = 1$ and $\cos(\pi) = -1$. Since $C^2 + S^2 = 1$ we have that $C \in [-1, 1]$. The intermediate value theorem gives some $\theta \in [0, \pi]$ so that $\cos(\theta) = C$. Note that $\cos(-\theta) = C$ as well. We deduce that $\sin^2(\theta) = S^2$. Taking square roots (and replacing θ by $-\theta$ if needed) proves that θ lies in $\textsc{Arg}(z)$.

Since cosine and sine are 2π–periodic, we deduce that $\theta + 2\pi k$ lies in $\textsc{Arg}(z)$ for all $k \in \mathbb{Z}$. Suppose that η is an element of $\textsc{Arg}(z)$. Using the addition law we have the following.

$$\begin{aligned}\cos(\theta - \eta) &= \cos(\theta)\cos(-\eta) - \sin(\theta)\sin(-\eta)\\ &= \cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)\\ &= C^2 + S^2\\ &= 1\end{aligned}$$

Thus $\theta - \eta$ lies in $2\pi\mathbb{Z}$, as desired. □

Solution 1.6. Suppose that $z = x + iy$. Suppose that $w = a + ib$. So $zw = xa - yb + i(xb + ay)$. We now compute as follows:

$$\begin{aligned}|zw|^2 &= (xa - yb)^2 + (xb + ya)^2\\ &= x^2a^2 + y^2b^2 - 2xyab + x^2b^2 + y^2a^2 + 2xyab\\ &= x^2a^2 + x^2b^2 + y^2a^2 + y^2b^2\\ &= (x^2 + y^2)(a^2 + b^2)\\ &= |z|^2|w|^2\end{aligned}$$

Taking the positive square root of both sides gives $|zw| = |z||w|$, as desired.

By the first part of Exercise 1.4 we may choose θ and η to be arguments of z and w respectively. By the second part of Exercise 1.4 it suffices to prove that $\theta + \eta$ is an argument of zw. We have the following polar forms:

$$z = |z|(\cos(\theta) + i\sin(\theta)) \qquad w = |w|(\cos(\eta) + i\sin(\eta))$$

Taking their product, and applying the addition rule, gives:

$$zw = |z||w|(\cos(\theta + \eta) + i\sin(\theta + \eta))$$

By the above we have $|z||w| = |zw|$. Thus $\theta + \eta$ is an argument of zw, as desired. □

Solution 1.7. Suppose that $z = x + iy$ and $w = r + is$. We compute as follows:

$$\begin{aligned}
&|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta) \\
&\quad = |z|^2 + |w|^2 - 2|z||w|(\cos(\theta)\cos(\eta) + \sin(\theta)\sin(\eta)) \\
&\quad = |z|^2 + |w|^2 - 2|z||w|\left(\frac{x}{|z|}\frac{r}{|w|} + \frac{y}{|z|}\frac{s}{|w|}\right) \\
&\quad = |z|^2 + |w|^2 - 2(xr + ys) \\
&\quad = x^2 + y^2 + r^2 + s^2 - 2(xr + ys) \\
&\quad = x^2 - 2xr + r^2 + y^2 - 2ys + s^2 \\
&\quad = (x - r)^2 + (y - s)^2 \\
&\quad = |z - w|^2
\end{aligned}$$

□

Solution 1.9. Suppose that a, b, c, and d are complex numbers. Suppose that f and g are similarities with $f(z) = az+b$ and $g(z) = cz+d$. Then $f \circ g(z) = a(cz + d) + b = acz + (ad + b)$. Since a and c are non-zero, so is ac. Thus $f \circ g$ is again a similarity.

We obtain an identity by taking $a = 1$ and $b = 0$. We make g an inverse for f by setting $c = 1/a$ and $d = -b/a$. The group operation is associative because function composition is associative.

Suppose that $|a| = |c| = 1$. Thus $|ac| = 1$ and $f \circ g$ is a congruence. Also $|1/a| = 1$, so the inverse of f is a congruence.

More generally, if the condition on a is closed under multiplication and inversion, and b is general or zero, then we obtain a subgroup. □

Solution 1.10. If $w = 0$ then $|z - w| < |z|$ becomes $|z| < |z|$, a contradiction. So $w \neq 0$. In particular $\textsc{Arg}(w)$ is well-defined.

Suppose that $\theta \in \textsc{Arg}(z)$ is the given argument of z. Fix any argument $\eta' \in \textsc{Arg}(w)$. Applying the cosine law we have

$$|z|^2 + |w|^2 - 2|z||w|\cos(\theta - \eta') < |z|^2$$

Simplifying we find

$$\frac{|w|}{2|z|} < \cos(\theta - \eta')$$

Thus $\cos(\theta - \eta')$ is strictly positive. So, for some $k \in \mathbb{Z}$, we have $|\theta - \eta' - 2\pi k| < \pi/2$. By Exercise 1.4 we have $\eta = \eta' + 2\pi k \in \textsc{Arg}(w)$. Thus $|\theta - \eta| < \pi/2$, as desired.

Suppose now that $\eta'' \in \textsc{Arg}(w)$ also satisfies $|\theta - \eta''| < \pi/2$. Thus

$$\begin{aligned}
|\eta - \eta''| &= |\eta - \theta + \theta - \eta''| \\
&\leq |\eta - \theta| + |\theta - \eta''| \\
&< \pi/2 + \pi/2 \\
&= \pi
\end{aligned}$$

But the difference $\eta - \eta''$ is a multiple of 2π (Exercise 1.4). Thus $\eta = \eta''$, as desired. □

Solution 2.4. For any point $p \in U$ we define U_p as follows: U_p is the set of points of U that can be reached from p by axis-aligned paths. Now, discs are path connected by axis-aligned paths. So, by concatenating paths, if $B \subset U$ is an open disc that meets U_p then B is contained in U_p. Thus U_p is open (a union of discs) and non-empty (it contains p).

Fix now some p in U. Define $V_p = U - U_p$. Note that $V_p = \cup_{q \in V_p} U_q$. Thus V_p is open. So $U = U_p \cup V_p$ is a union of open sets which are disjoint. Thus one of U_p or V_p is empty. So V_p is empty and we are done. □

Solution 2.5. We must show that $U^\times$ is open, non-empty, and connected.

It is open because it is the intersection of open sets: $U^\times = U \cap (\mathbb{C} - \{z_0\})$. It is non-empty because, if not, then $U = \{z_0\}$ which is not open.

Suppose that p and q are points of $U^\times$. Suppose that $\beta\colon [a,b] \to U$ is an axis-aligned path that connects p to q; this exists by Exercise 2.4. Suppose that $B = \overline{B(z_0;\delta)}$ is a disc contained in U. Each component of $\beta \cap B$ meeting z_0 can be replaced by an axis-aligned path in B that avoids z_0. Thus $U^\times$ is path-connected. □

Solution 2.6. If $U = \mathbb{C}$ then any constant $\epsilon > 0$ works. Suppose instead that U is a proper subset of $\mathbb{C}$. Suppose that q lies in $\mathbb{C} - U$. Since distance is continuous and P is compact, there is some point $p \in P$ which is as close as possible to q. Thus writing $d(q,P) = |q-p|$ is well-defined. Suppose, for a contradiction, that (q_n) is a sequence in $\mathbb{C} - U$ so that $d(q_n, P)$ tends to zero. Let p_n be a point of P closest to q_n. Since P is compact, the sequence (q_n) is bounded, so has a convergent subsequence, which we pass to. Since $\mathbb{C} - U$ is closed, the limit q_∞ lies in $\mathbb{C} - U$. We pass to a further subsequence to ensure that (p_n) also converges say to p_∞ in P. But then $|q_\infty - p_\infty| = 0$. Thus $q_\infty = p_\infty$. We deduce that P and $\mathbb{C} - U$ intersect, a contradiction. □

Solution 2.10. Suppose that z lies in $\mathbb{C}^\times$. Since $|z|$ is positive it has a positive square root $\sqrt{|z|}$. Take any $\theta \in \textsc{Arg}(z)$. We set $\zeta = \sqrt{|z|}e^{i\theta/2}$. So $\zeta^2 = (\sqrt{|z|}e^{i\theta/2})^2 = |z|e^{i\theta} = z$; this gives one square root of z.

Note that $-\zeta \neq \zeta$ because $\zeta \neq 0$. Since $(-1)^2 = 1$ we have now found another square root; we must show that there are no more.

Fix θ and ζ as above. Suppose that u is some square root of z. Suppose that η is an argument of u. So, by Exercise 1.6, we have that 2η is an argument of z. Thus, by Exercise 1.4, we have that $\theta - 2\eta = 2\pi k$ for some k in $\mathbb{Z}$. We deduce that $u = \pm\zeta$ as k is even or odd.

We now produce a branch of Sqrt. First we define a function $f\colon B \to \mathbb{C}$. Fix any $\theta_0 \in \textsc{Arg}(z_0)$. Suppose that w lies in B. By

Exercise 1.10 there is a unique $\eta \in \text{ARG}(w)$ so that $|\theta_0 - \eta| < \pi/2$. We define $f(w) = |w|^{1/2} e^{i\eta/2}$. By the above, $f(w)$ is a square root of w.

We now must prove f is continuous. Fix w_0 in B. Let $\eta_0 \in \text{ARG}(w_0)$ be the angle selected in the previous paragraph. Pick any $\epsilon \in (0, \pi/2)$. Fix $\delta > 0$ so that $B(w_0; \delta) \subset B$ and also so that δ satisfies Lemma 1.11. So, for any $w_1 \in B(w_0; \delta)$, we have $w_1 \neq 0$ and also have a unique $\eta_1 \in \text{ARG}(w_1)$ with $|\eta_0 - \eta_1| < \epsilon$. We deduce from the triangle inequality and Exercise 1.4 that η_1 is the argument for w_1 picked in the previous paragraph. The continuity of f now follows from the continuity of the real square root, the real sine, and the real cosine. Thus f is a branch of SQRT.

Suppose that g is another branch of SQRT. So g is continuous and never zero. So f/g is continuous and squares to give the function that is identically one on B. Thus for all $z \in B$ we have $f(z)/g(z) = \pm 1$. Since f/g is continuous it is constant. Thus $-f$ is the only other branch of SQRT. □

Solution 2.18. We have already dealt with the case $k = -1$ in Example 2.17. So suppose that $k \neq -1$. We compute as follows:

$$\begin{aligned}
\int_\gamma z^k \, dz &= \int_0^{2\pi} e^{ki\theta} \cdot i e^{i\theta} \, d\theta \\
&= i \int_0^{2\pi} e^{(k+1)i\theta} \, d\theta \\
&= \frac{i}{(k+1)i} \left[e^{(k+1)i\theta} \right]_0^{2\pi} \\
&= \frac{1}{k+1}(1-1) \\
&= 0
\end{aligned}$$

Here we used the fundamental theorem of calculus for the real-valued cosine and sine (not for the complex exponential). □

Solution 2.20. We reparametrise γ to instead use the domain $[-\pi, \pi]$. We now compute as follows:

$$\begin{aligned}\int_\gamma f\,dz &= \int_\gamma \frac{dz}{z} \\ &= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{2+e^{i\theta}}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{e^{i\theta}}{2+e^{i\theta}}\frac{2+e^{-i\theta}}{2+e^{-i\theta}}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{2e^{i\theta}+1}{5+4\cos(\theta)}\,d\theta \\ &= i\int_{-\pi}^{\pi} \frac{2\cos(\theta)+2i\sin(\theta)+1}{5+4\cos(\theta)}\,d\theta\end{aligned}$$

We consider the real and imaginary parts separately.

The real part equals:

$$-2\int_{-\pi}^{\pi} \frac{\sin(\theta)}{5+4\cos(\theta)}\,d\theta$$

This is a definite integral with odd integrand and symmetric domain of integration. Thus the integral vanishes.

The imaginary part (after dividing by i) equals:

$$\int_{-\pi}^{\pi} \frac{1+2\cos(\theta)}{5+4\cos(\theta)}\,d\theta$$

We substitute using

$$\cos(\theta) = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad d\theta = \frac{2}{1+t^2}\,dt$$

The imaginary part becomes:

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{1+2\cos(\theta)}{5+4\cos(\theta)}\,d\theta \\ &\quad= \int_{-\infty}^{\infty} \frac{(1+t^2)+2(1-t^2)}{5(1+t^2)+4(1-t^2)}\,\frac{2}{1+t^2}\,dt \\ &\quad= \int_{-\infty}^{\infty} \frac{3-t^2}{9+t^2}\,\frac{2}{1+t^2}\,dt\end{aligned}$$

Since the integrand is even, it suffices to show that the integral over $(0,\infty)$ vanishes. Comparing with $\int_1^\infty dt/t^2$ we find that the (improper) integral converges. The integrand changes sign at $t=\sqrt{3}$. This suggests

we use the substitution $t = 3/u$ (so $dt = -3du/u^2$):

$$\begin{aligned}
&\int_0^\infty \frac{3-t^2}{9+t^2}\,\frac{2}{1+t^2}\,dt \\
&\quad = \int_\infty^0 \frac{3-9/u^2}{9+9/u^2}\,\frac{2}{1+9/u^2}\,\frac{-3}{u^2}\,du \\
&\quad = \int_\infty^0 \frac{3/u^2-1}{1+1/u^2}\,\frac{2}{1+9/u^2}\,\frac{1}{u^2}\,du \\
&\quad = \int_\infty^0 \frac{3-u^2}{u^2+1}\,\frac{2}{u^2+9}\,du
\end{aligned}$$

Thus the integral equals its negative, and so vanishes.

As a sanity check we numerically estimate the integral over $(0, \sqrt{3})$ and over $(\sqrt{3}, \infty)$.

```
sage: var("t"); f = 2*(3 - t^2) / ((9 + t^2)*(1 + t^2))
sage: numerical_integral(f, 0, sqrt(3))
(0.5235987755982988, 5.813114163347719e-15)
sage: numerical_integral(f, sqrt(3), Infinity)
(-0.5235987755982999, 2.302746825788907e-10)
```

In each answer, the first entry is an estimate of the integral; the second is an estimate of the error. (Our answer raises a further question: why is the numerical integral so very close to $\pi/6$?) □

Solution 2.23. This follows from Lemma 2.22 and Exercise 2.18. □

Solution 2.24. Fix any $\epsilon > 0$. Since U is open and f is continuous, there is some $\eta > 0$ so that $B(w;\eta) \subset U$ and so that $z \in B(w;\eta)$ implies $|f(z) - f(w)| \le \epsilon$. Fix any $0 < \delta < \eta$ and compute as follows:

$$\begin{aligned}
|I_\delta - f(w)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - f(w)\right| && \text{definition of } I_\delta \\
&= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}\,dz\right| && \text{Example 2.19} \\
&= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(w)}{z-w}\,dz\right| && \text{Lemma 2.22} \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(w+\delta e^{i\theta})-f(w)}{\delta e^{i\theta}}\delta i e^{i\theta}\,d\theta\right| && \text{Definition 2.16} \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} (f(w+\delta e^{i\theta})-f(w))\,d\theta\right| && \text{simplify} \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|(f(w+\delta e^{i\theta})-f(w))\right|\,d\theta && \text{triangle inequality} \\
&\le \epsilon
\end{aligned}$$

□

Solution 2.26. By Lemma 2.25, and induction, the problem reduces to the case where γ has only one piece. We now write $\gamma(t) = u(t)+iv(t)$. So $\gamma'(t) = u'(t) + iv'(t)$. Linearity with respect to the integrand, and the real fundamental theorem of calculus, complete the proof. □

Solution 2.32. We use the notation $\gamma = (\gamma_x, \gamma_y)$.

Suppose that δ preserves *orientation*: that is, $\delta(c) = a$ and $\delta(d) = b$. Then δ' is non-negative. Suppose that δ reverses orientation: that is, $\delta(c) = b$ and $\delta(d) = a$. Then δ' is non-positive.

Suppose first that δ preserves orientation. Then we compute as follows.

$$\begin{aligned}
L(\gamma \circ \delta) &= \int_{\gamma\circ\delta} \sqrt{dx^2 + dy^2} \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2(\delta'(u))^2 + (\gamma_y'(\delta(u)))^2(\delta'(u))^2}\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot |\delta'(u)|\, du \\
&= \int_c^d \sqrt{(\gamma_x'(\delta(u)))^2 + (\gamma_y'(\delta(u)))^2} \cdot \delta'(u)\, du \\
&= \int_a^b \sqrt{(\gamma_x'(t))^2 + (\gamma_y'(t))^2}\, dt \\
&= \int_\gamma \sqrt{dx^2 + dy^2} \\
&= L(\gamma)
\end{aligned}$$

If δ reverses orientation, then the two minus signs cancel, and we obtain the same arc-length. □

Solution 2.35. Induction (and the triangle inequality) reduces us to the case where γ has only one piece. The proof where there is only one piece is the same as the usual proof using Riemann sums. □

Solution 2.37. We apply Exercise 2.35, Definition 2.34, and Definition 2.29 to obtain:

$$\left|\int_\gamma f\, dz\right| \leq \int_\gamma |f\, dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leq M \cdot L(\gamma) \qquad \square$$

Solution 3.5. Suppose that $\gamma(s) = x(s) + iy(s)$. We suppress the variable s to obtain $\gamma = x + iy$ and $\gamma' = x' + iy'$. We may also write $r = \sqrt{x^2 + y^2}$. Thus $r'/r = (xx' + yy')/(x^2 + y^2)$. The real fundamental theorem of calculus gives $\theta' = (xy' - yx')/(x^2 + y^2)$. We compute:

$$\frac{\gamma'}{\gamma} = \frac{x' + iy'}{x + iy} = \frac{x' + iy'}{x + iy}\frac{(x - iy)}{(x - iy)} = \frac{xx' + yy' + i(xy' - yx')}{x^2 + y^2} = \frac{r'}{r} + i\theta'$$

Recall that $h = re^{i\theta}/\gamma$. We differentiate:

$$h' = \frac{r'e^{i\theta} + re^{i\theta}i\theta'}{\gamma} - \frac{re^{i\theta}\gamma'}{\gamma^2} = \frac{re^{i\theta}}{\gamma}\left[\frac{r'}{r} + i\theta' - \frac{\gamma'}{\gamma}\right] = 0 \qquad \square$$

Solution 4.3. Fix z_0 in U. Fix $\epsilon > 0$. Since domains are open, and from Definition 4.2, there is a δ' so that $B' = B(z_0; \delta')$ is contained in U and, for all $z \in B' - \{z_0\}$ we have:

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

We take

$$\delta = \min(\delta', \epsilon/(|f'(z_0)| + \epsilon))$$

Let $B = B(z_0; \delta)$. Note that $B \subset B'$.

Suppose now that z lies in B. If $z = z_0$ then $|f(z) - f(z_0)| = 0 < \epsilon$ and we are done. Suppose instead that $z \neq z_0$. Then we can clear the denominator of the above inequality to obtain:

$$|(f(z) - f(z_0)) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|$$

Thus

$$\begin{aligned} |f(z) - f(z_0)| &< |f'(z_0)||z - z_0| + \epsilon|z - z_0| \\ &< (|f'(z_0)| + \epsilon)\delta \\ &\leq \epsilon \end{aligned}$$

Thus f is continuous at z_0. $\square$

Solution 4.5. Since f is continuous (by Exercise 4.3) we have that ρ is continuous away from z_0. The limit as z tends to z_0 is zero, by construction and since f is holomorphic. Thus ρ is also continuous at z_0. $\square$

Solution 4.8. The proof is, symbol-by-symbol, the same as the real one-variable case. $\square$

Solution 4.9. Fix any w_0 and w_1 in U. Suppose that $\beta\colon [a, b] \to U$ is an axis-aligned contour from w_0 to w_1 (Exercise 2.4). Induction reduces us to the case where β is a single segment.

Fix $\epsilon > 0$. Recall that f is holomorphic and f' vanishes. So, for z' in β we have some $\delta' > 0$ so that if $z \in B(z'; \delta')$, then $|f(z) - f(z')| \leq \epsilon|z - z'|$. Such discs cover β, which is compact. So there is a finite subcover. We now walk along β and build sequences of points $(a_k)_{k=0}^N$ and $(b_k)_{k=1}^N$ as follows:

- $w_0 = a_0$ and $a_N = w_1$,
- b_k is the centre of $B_k = B(b_k; \delta_k)$ a disc as above,
- $|w_0 - a_{k-1}| < |w_0 - b_k| < |w_0 - a_k|$ for $k \geq 1$, and
- $|a_{k-1} - b_k|, |b_k - a_k| < \delta_k$.

Thus we have:

- $|f(a_{k-1}) - f(b_k)| \leq \epsilon |a_{k-1} - b_k|$, and
- $|f(b_k) - f(a_k)| \leq \epsilon |b_k - a_k|$.

We compute as follows, applying the triangle inequality:

$$\begin{aligned} |f(w_0) - f(w_1)| &= \left| \sum f(a_{k-1}) - f(b_k) + \sum f(b_k) - f(a_k) \right| \\ &\leq \sum |f(a_{k-1}) - f(b_k)| + \sum |f(b_k) - f(a_k)| \\ &\leq \sum \epsilon |a_{k-1} - b_k| + \sum \epsilon |b_k - a_k| \\ &= \epsilon |w_0 - w_1| \end{aligned}$$

Since ϵ was arbitrary, we deduce that $f(w_0) = f(w_1)$, as desired. □

Solution 4.15. Applying the geometric series we find

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \dots$$

Thus $R = 1$. □

Solution 4.16. We may factor the quadratic into linear terms, realise f as a (scaled) sum of their reciprocals, apply the geometric series, and add coefficients (appealing to Theorem 4.14). Thus f has a power series about zero. Suppose that $(a_n)_n$ are the coefficients. Since $f(z) \cdot (1 + z + z^2) = 1$ we can multiply out, equate coefficients, and find:

$$a_0 = 1, \quad a_1 + a_0 = 0, \quad a_2 + a_1 + a_0 = 0$$

and in general $a_{n+2} + a_{n+1} + a_n = 0$ for $n \geq 0$. By induction, we have

$$a_n = \begin{cases} 1, & \text{if } n = 0 \pmod 3 \\ -1, & \text{if } n = 1 \pmod 3 \\ 0, & \text{if } n = 2 \pmod 3 \end{cases}$$

Thus the radius of convergence is one. □

Solution 4.19. Suppose that $\textsc{Sin}(z) = 0$. Thus $\textsc{Cos}(z) = \pm 1$.

Suppose that $\textsc{Cos}(z) = 1$. Thus $\textsc{Exp}(iz) = 1$. Suppose that $z = x + iy$. We now compute using the properties of complex exponential.

$$\begin{aligned} 1 &= \textsc{Exp}(iz) \\ &= \textsc{Exp}(ix - y) \\ &= \textsc{Exp}(ix)\,\textsc{Exp}(-y) \\ &= (\textsc{Cos}(x) + i\,\textsc{Sin}(x))\,\textsc{Exp}(-y) \\ &= (\cos(x) + i\sin(x))\exp(-y) \\ &= \cos(x)\exp(-y) + i\sin(x)\exp(-y) \end{aligned}$$

Equating real and imaginary parts we find that

$$1 = \cos(x)\exp(-y) \quad \text{and} \quad 0 = \sin(x)\exp(-y)$$

Since the exponential has no roots, we find that $\sin(x) = 0$. Thus $x = \pi k$ for some $k \in \mathbb{Z}$. Thus $\cos(x) = \pm 1$. However, $\exp(-y)$ is

positive. We deduce that $\cos(x) = 1$ and $y = 0$. Thus $z = x + iy = \pi k$, with k even.

When $\textsc{Cos}(z) = -1$ we find instead that k is odd.

The zeros of Cos are handled similarly (or by using an identity). □

Solution 4.20. Note that $\textsc{Exp}(x + iy) = \textsc{Exp}(x)\,\textsc{Exp}(iy) = e^x e^{iy}$. Recall that e^x is a positive real number, while e^{iy} is a "direction": a point on the unit circle. If we vary x while fixing y then we obtain a radial arc with direction e^{iy}. If we fix x while varying y then we obtain an arc of the circle, centred on the origin, with radius e^x.

We now take the image of the rectangle R. The horizontal sides both go to the line segment $[a, b]$ in the real axis. The left vertical side goes to the circle $C(0; a)$; the right vertical side goes to the circle $C(0; b)$. The interior of R is sent to the inside of the resulting (non-simple) contour. □

Solution 5.5. Similarities preserve convexity. So it suffices to prove this for the unit disc $\mathbb{D} = B(0; 1)$ and its closure $\overline{\mathbb{D}}$. Suppose that p and q are points in $\mathbb{D}$. Suppose that $t \in [0, 1]$. We must show that $(1 - t)p + tq$ lies in $\mathbb{D}$.

Since p and q lie in $\mathbb{D}$ there is some $r < 1$ so that both $|p|$ and $|q|$ are at most r. We now apply the triangle inequality to find:

$$|(1 - t)p + tq| \leq (1 - t)|p| + t|q| \leq (1 - t)r + tr = r < 1$$

For $\overline{\mathbb{D}}$ we replace the condition $r < 1$ by the weaker condition $r \leq 1$. □

Solution 5.6. Suppose that H is the convex hull of S. Suppose that p and q are points of H. Suppose that K is a convex set containing S. So $H \subset K$. So p and q are points of K. Thus $[p, q]$ is a subset of K. Thus $[p, q]$ is a subset of H, as desired. □

Solution 5.12. We recall (without proof) a few properties of the cross-product:

$(tu) \times v = t(u \times v)$	scaling for t real
$au \times av = \lvert a\rvert^2(u \times v)$	scaling for a complex
$u \times (v + w) = u \times v + u \times w$	distributivity
$v \times u = -u \times v$	anti-symmetry

We also have that $u \times v = 0$ if and only if the origin, u, and v lie on a line.

The first claim of the exercise follows from the symmetry of the definition of $\mathcal{O}$. The second claim follows from the anti-symmetry of the cross product.

We now prove the third claim. If $u \times v = v \times w = w \times u = 0$ then there is a line through the origin containing all of u, v, and w. In

this case we are done. So, as the other cases are similar, suppose that $u \times w \neq 0$. So u and w are a basis for $\mathbb{C}$ (thought of as a vector space over $\mathbb{R}$). So we have $v = su + tw$ for some s and t real. We compute as follows:

$$\begin{aligned}\mathcal{O}(u,v,w) &= u \times v + v \times w + w \times u \\ &= u \times (su + tw) + (su + tw) \times w + w \times u \\ &= t(u \times w) + s(u \times w) + w \times u \\ &= (s + t - 1)(u \times w)\end{aligned}$$

So $\mathcal{O}(u,v,w) = 0$ if and only if $s = 1 - t$ if and only if $v = (1-t)u + tw$. That is, if and only if u, v, and w lie on a line.

The fourth claim states that $\mathcal{O}$ is invariant under rotation and translation, and transforms as the square under scaling. To prove it, we compute as follows:

$$\begin{aligned}&\mathcal{O}(au + b, av + b, aw + b) \\ &\quad= \quad (au + b) \times (av + b) \\ &\qquad + (av + b) \times (aw + b) \\ &\qquad + (aw + b) \times (au + b) \\ &\quad= \quad au \times av + au \times b + b \times av + b \times b \\ &\qquad + av \times aw + av \times b + b \times aw + b \times b \\ &\qquad + aw \times au + aw \times b + b \times au + b \times b \\ &\quad= |a|^2 \, (u \times v + v \times w + w \times u) \\ &\quad= |a|^2 \, \mathcal{O}(u,v,w)\end{aligned}$$ □

Solution 5.15. An "informal" definition and proof comes from drawing a picture.

More formally we have the following. Suppose that u and v are any points on L with v after u. Suppose that w is any complex number not on L. Then w is *to the left of* L if $\mathcal{O}(u,v,w)$ is positive. (One must check that this is well defined.)

Suppose now that L is the line containing $[p,q]$. So, in particular, the vertex r of $T = T(p,q,r)$ is to the left of L. Note that $L \cap T = [p,q]$. This, plus the continuity of $\mathcal{O}$, proves that every point of $T - [p,q]$ is to the left of L. □

Solution 6.6. We break $I(\partial T)$ into pieces using its definition and Lemma 2.25:

$$I(\partial T) = I([p,q]) + I([q,r]) + I([r,p])$$

Each term on the right can be broken in two using the midpoints, Lemma 2.25, and the first part of Lemma 2.27:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q]) \\ & + I([q, p']) + I([p', r]) \\ & + I([r, q']) + I([q', p]) \end{aligned}$$

We rearrange terms:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([q', p]) \\ & + I([r', q]) + I([q, p']) \\ & + I([p', r]) + I([r, q']) \end{aligned}$$

We add and subtract terms:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q']) + I([q', p]) \\ & + I([r', q]) + I([q, p']) + I([p', r']) \\ & + I([q', p']) + I([p', r]) + I([r, q']) \\ & - I([r', q']) - I([p', r']) - I([q', p']) \end{aligned}$$

We apply the second part of Lemma 2.27 and rearrange:

$$\begin{aligned} I(\partial T) = & I([p, r']) + I([r', q']) + I([q', p]) \\ & + I([q, p']) + I([p', r']) + I([r', q]) \\ & + I([r, q']) + I([q', p']) + I([p', r]) \\ & + I([p', q']) + I([q', r']) + I([r', p']) \end{aligned}$$

We now use Lemma 2.25 and the definitions:

$$I(\partial T) = I(\partial P) + I(\partial Q) + I(\partial R) + I(\partial S)$$

This proves the claim. □

Solution 6.7. Since the triangles are compact, and nested, their intersection contains at least one point (Cantor intersection theorem). Since the diameters tend to zero, the intersection contains at most one point. □

Solution 6.8. We cut R along a diagonal into a pair of positively oriented triangles S and T. By Lemma 6.3 the integrals of f over ∂S and over ∂T both vanish. Adding these integrals, and applying Lemmas 2.25 and 2.27, we find that the integral of f over ∂R also vanishes. □

Solution 7.3. With notation as above, we have $\partial P = \cup \beta_k$. Let L_k be the line in $\mathbb{C}$ containing β_k. Suppose that R is a component of $\mathbb{C} - \cup L_k$. Note that R is convex. If R contains points $p \in P$ and $q \notin P$ then we deduce that ∂P meets the interior of R, a contradiction. We deduce that R is either contained in P or is disjoint from P. Suppose that R is contained in P. Then $\overline{R}$ is a finite-sided convex polygon. Fix some

vertex v of ∂R; we connect v to all of the other vertices of ∂R by line segments. This realises $\overline{R}$ as a union of triangles. Since P° is dense in P, the union of these triangles (as R ranges over the pieces of $P - \cup L_k$) gives the desired decomposition of P.

(There are ways to decompose P into triangles without adding new vertices. For one early proof, see [7, page 46].) □

Solution 7.4. Let (T_ℓ) be the decomposition of P given in Exercise 7.3, all oriented positively. By Lemma 6.3 the integral of f about ∂T_ℓ is zero. Thus

$$\sum_\ell \int_{\partial T_\ell} f\,dz = 0$$

On the other hand, where two distinct triangles T_ℓ and $T_{\ell'}$ intersect in more than a vertex, they intersect in a single line segment; by Exercise 5.15, the boundaries of T_ℓ and $T_{\ell'}$ give this segment opposite orientations. Thus, applying Lemma 2.25 and Lemma 2.27, we have

$$\sum_\ell \int_{\partial T_\ell} f\,dz = \int_{\partial P} f\,dz$$

So the latter vanishes and we are done. □

Solution 7.10. By Example 2.17, the integral of f about the unit circle does not vanish. Thus, by Lemma 7.9, the function f does not have a primitive in $\mathbb{C}^\times$. □

Solution 8.5. They are $\sigma^3|[v_1, v_2, v_3]$, $\sigma^3|[v_0, v_2, v_3]$, $\sigma^3|[v_0, v_1, v_3]$, and $\sigma^3|[v_0, v_1, v_2]$. □

Solution 8.9. Since the boundary operators are homomorphisms, it suffices to fix a single singular two-simplex $\sigma^2\colon \Delta^2 \to X$ and then to prove that $\partial_1\partial_2\sigma^2 = 0$. We compute as follows:

$$\begin{aligned}
(\partial_1 \circ \partial_2)\sigma^2 &= \partial_1(\partial_2\sigma^2) \\
&= \partial_1(\sigma^2|[v_1, v_2] - \sigma^2|[v_0, v_2] + \sigma^2|[v_0, v_1]) \\
&= \partial_1(\sigma^2|[v_1, v_2]) - \partial_1(\sigma^2|[v_0, v_2]) + \partial_1(\sigma^2|[v_0, v_1]) \\
&= (\sigma^2|[v_2] - \sigma^2|[v_1]) \\
&\quad - (\sigma^2|[v_2] - \sigma^2|[v_0]) \\
&\quad + (\sigma^2|[v_1] - \sigma^2|[v_0]) \\
&= 0
\end{aligned}$$

□

Solution 8.16. Along $[v_0, v_2]$ we have $x_1 = 0$ and $x_2 \in [0, 1]$ (with that orientation). So $c|[v_0, v_2]$ is the function $x_2 \mapsto 1 - x_2$.

Along $[v_1, v_2]$ we have $x_0 = 0$, we have $x_1 + x_2 = 1$, and we have $x_2 \in [0, 1]$ (with that orientation). We deduce that $x_1 = 1 - x_2$. So $c|[v_1, v_2]$ is, again, the function $x_2 \mapsto 1 - x_2$. □

Solution 8.17. We define $\sigma^2, \tau^2 \colon \Delta^2 \to X$ by

$$\sigma^2(x_0, x_1, x_2) = \sigma^1((x_0 - x_1 + 1)/2, (x_1 - x_0 + 1)/2)$$
$$\tau^2(x_0, x_1, x_2) = \tau^1((x_0 - x_1 + 1)/2, (x_1 - x_0 + 1)/2)$$

Note that $\sigma^2|[v_0, v_1] = \sigma^1$ and $\tau^2|[v_0, v_1] = \tau^1$. Also, $\sigma^2|[v_1, v_2] = \tau^2|[v_0, v_2]$, while $\sigma^2|[v_0, v_2] = \tau^2|[v_1, v_2]$. Taking care with signs, we find that $\partial(\sigma^2 + \tau^2) = \sigma^1 + \tau^1$, as desired. □

Solution 8.19. We take $v = (a, b)$ and note that $a + b = 1$. We define $\sigma^2 \colon \Delta^2 \to X$ by

$$\sigma^2(x_0, x_1, x_2) = \sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2)$$

First, suppose that $x_1 = 0$. Then we have $x_0 + x_2 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a + bx_0 - a(1 - x_0), b - b(1 - x_2) + ax_2) \\ &= \sigma^1(x_0, x_2)\end{aligned}$$

Thus $\sigma^2|[v_0, v_2] = \sigma^1$.

Next, suppose that $x_0 = 0$. Then we have $x_1 + x_2 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a - ax_2, b + ax_2) \\ &= \sigma^1(a(1 - x_2), b + a(1 - x_1)) \\ &= \sigma^1(ax_1, 1 - ax_1)\end{aligned}$$

Thus $\sigma^2|[v_1, v_2] = \sigma^1|[v, v_1]$.

Finally, suppose that $x_2 = 0$. Then we have $x_0 + x_1 = 1$. In this case we find that

$$\begin{aligned}\sigma^1(a + bx_0 - ax_2, b - bx_0 + ax_2) &= \sigma^1(a + bx_0, b - bx_0) \\ &= \sigma^1(a + b(1 - x_1), b(1 - x_0)) \\ &= \sigma^1(1 - bx_1, bx_1)\end{aligned}$$

Thus $\sigma^2|[v_0, v_1] = \sigma^1|[v_0, v]$. □

Solution 8.21. Suppose that σ^2 is a singular two-simplex. It suffices to show that $\partial_2(\textsc{Mid}_2(\sigma^2)) = \textsc{Mid}_1(\partial_2(\sigma^2))$. The proof is given by Figure 6.4, but reversing the arrows in the triangle S and recalling the minus sign in the definition of $\textsc{Mid}_2$. □

Solution 9.3. Note that γ is a closed contour in $\mathbb{C}^\times$. Note that $f(z) = 1/z$ is holomorphic in $\mathbb{C}^\times$. Recall from Example 2.17 that $\int_\gamma dz/z \neq 0$. Now apply the contrapositive of Theorem 9.2: since the contour integral does not vanish γ is not a one-boundary. □

Solution 10.4. Directly from the definitions we have $\partial \circ f_*(\sigma) = f_* \circ \partial(\sigma)$.

Suppose that a is a one-cycle in U. Take $c = f_*(a)$. So $\partial c = \partial f_*(a) = f_*\partial a = 0$. Thus f_* sends one-cycles in U to one-cycles in V.

Suppose that a is a one-boundary in U. Take $c = f_*(a)$. Suppose that b is a two-chain in U with $\partial b = a$. Take $d = f_*(b)$. So we have $\partial d = \partial f_*(b) = f_*\partial b = f_*(a) = c$. So f_* sends one-boundaries in U to one-boundaries in V. We now define $[f]\colon H_1(U) \to H_1(V)$ by $[f][c] = [f_*(c)]$. From the above this is well-defined and is a homomorphism.

The same holds for g replacing f. Since f and g are inverses, it follows from the definitions that $[f]$ and $[g]$ are inverses. □

Solution 14.6. This follows from the definitions and the ML–inequality (Lemma 2.36). Here is the computation:

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\
&\le \frac{1}{2\pi} \int_C \left| \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\
&\le \frac{1}{2\pi} \int_C \frac{M}{r^{n+1}}\, |dz| \\
&\le \frac{M}{2\pi r^{n+1}} 2\pi r \\
&\le \frac{M}{r^n}
\end{aligned}$$ □

Solution 14.8. Set $F_N(w) = \sum_{n=0}^{N} a_n (w - z_0)^n$. We repeat the computation, replacing F by F_N.

$$\begin{aligned}
F_N(w) &= \sum_{n=0}^{N} a_n (w-z_0)^n && \text{definition } F_N(w) \\
&= \sum \left[\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right] (w-z_0)^n && \text{definition } a_n \\
&= \frac{1}{2\pi i} \int \left[\sum \frac{f(z)}{(z-z_0)^{n+1}} (w-z_0)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \left[\sum \left(\frac{w-z_0}{z-z_0} \right)^n \right] dz && \text{linearity} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} \left[\frac{1 - \left(\frac{w-z_0}{z-z_0} \right)^{N+1}}{1 - \left(\frac{w-z_0}{z-z_0} \right)} \right] dz && \text{geometric sum} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i} \int \frac{f(z)}{z-w} \left(\frac{w-z_0}{z-z_0} \right)^{N+1} dz && \text{simplify} \\
&= f(w) - \frac{1}{2\pi i} \int \frac{f(z)}{z-w} \left(\frac{w-z_0}{z-z_0} \right)^{N+1} dz && \text{Theorem 13.1}
\end{aligned}$$

Note that w lies inside of $B(z_0;r)$. Recall that C is the boundary of $\overline{B(z_0;r)}$. Thus

$$M' = \max\{|f(z)/(z-w)| : z \in C\}$$

is well-defined. Again with z in C we have that $\rho = |w-z_0|/|z-z_0|$ is constant and less than one. Rearranging and applying the ML–inequality (Lemma 2.36) we have:

$$|f(w) - F_N(w)| \leq \frac{1}{2\pi} M' \rho^{N+1} 2\pi r \leq M' r \rho^{N+1}$$

Note that M' and r are independent of N. Finally, ρ^{N+1} tends to zero as N tends to infinity. □

Solution 15.5. Recall that U is path-connected (Exercise 2.4). So, fix a path $\gamma\colon [0,1] \to U$ connecting z_0 to z_1. By Exercise 2.6 there is some $\epsilon > 0$ so that the open neighbourhood $N(\gamma;\epsilon)$ lies in U. Note that γ is continuous on a compact set, so it is uniformly continuous. So there is a $\delta > 0$ with the following property: if $s,t \in [0,1]$ have $|s-t| < \delta$ then $|\gamma(s) - \gamma(t)| < \epsilon/2$.

We now define a sequence of points $(w_k)_{k=0}^N$. Take $t_0 = 0$ and take $w_0 = \gamma(t_0) = z_0$. Suppose that we have obtained $t_\ell \in [0,1]$ and $w_\ell = \gamma(t_\ell)$, with $t_\ell < 1$. So we take $t_{\ell+1} \in [0,1]$ to be the maximum of $\gamma^{-1}(\overline{B(w_\ell;\epsilon/2)})$ and take $w_{\ell+1} = \gamma(t_{\ell+1})$. Since $\gamma(t_\ell) = w_\ell$, we have that t_ℓ lies in $\gamma^{-1}(\overline{B(w_\ell;\epsilon/2)})$. Thus $t_\ell < t_{\ell+1}$. If $t_{\ell+1} = 1$ then $w_{\ell+1} = z_1$. In this case we take $N = \ell + 1$ and the construction is complete.

Suppose instead that $t_{\ell+1} < 1$. In this case $|w_{\ell+1} - w_\ell| = \epsilon/2$. Thus $t_{\ell+1} - t_\ell \geq \delta$. This holds for all ℓ. We deduce that $\ell < 1/\delta$. In particular, the construction ends after at most $1 + 1/\delta$ steps. Thus we have produced $(w_k)_{k=0}^N$ in U so that

- $w_0 = z_0$,
- $w_N = z_1$,
- $|w_k - w_{k+1}| \leq \epsilon/2$, and
- $B(w_k;\epsilon)$ lies in U.

This completes the proof. □

Solution 15.12. Suppose that f (say) vanishes to infinite order at z_0. By Theorem 4.13 the series coefficients and the derivatives determine each other. So, taking derivatives and applying the product rule, we find that fg also vanishes to infinite order at z_0.

Suppose now that both $\textsc{Ord}(f,z_0)$ and $\textsc{Ord}(g,z_0)$ are finite. By Lemma 15.11 we can factor the corresponding powers of $(z-z_0)$ out of f and g. Exponents add when multiplying powers, so we are done. □

Index of symbols

$\square$, end of proof/solution, Lemma 1.11, 5
$\diamondsuit$, end of example/definition, Exercise 1.4, 3
$\textsc{Real}(z)$, real part, Section 1.2, 2
$\textsc{Imag}(z)$, imaginary part, Section 1.2, 2
$\bar{z}$, complex conjugate, Section 1.2, 2
$|z|$, magnitude, Section 1.2, 2
$B(z;r)$, open disc, Section 1.2, 2
$\overline{B(z;r)}$, closed disc, Section 1.2, 2
$B^{\times}(z;r)$, punctured disc, Section 1.2, 2
$\mathbb{C}^{\times}$, punctured plane, Section 1.2, 2
P°, interior, Section 1.2, 2
$\overline{P}$, closure, Section 1.2, 2
∂P, topological boundary, Section 1.2, 2
$C(z;r)$, circle, Section 1.2, 3
$\mathbb{D}$, open unit disc, Section 1.2, 3
$\mathbb{D}^{\times}$, punctured unit disc, Section 1.2, 3
S^1, unit circle, Section 1.2, 3
$\mathbb{H}$, upper-half plane, Section 1.2, 3
$N(Z;r)$, open neighbourhood, Section 1.2, 3
$\textsc{Arg}(z)$, arguments, Section 1.3, 3
γ, contour, Definition 2.12, 6
$\int_{\gamma} f\,dz$, contour integral, Definition 2.16, 7
$L(\gamma)$, arc-length, Definition 2.29, 10
$\textsc{Wind}(\gamma, z_0)$, winding number, Remark 3.8, 13
$D(\gamma)$, diameter, Section 5.16, 22
$T(p,q,r)$, triangle, Definition 5.7, 20
∂T, polygonal boundary, Definition 5.7, 20
$u \times v$, cross-product, Section 5.10, 21

References

[1] Lars V. Ahlfors. *Complex analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable. [1, 16, 27]

[2] Tom M. Apostol. *Mathematical analysis*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., second edition, 1974. [17]

[3] Leonhard Euler. *Mechanica sive Motus scientia analytice exposita*. Petropoli: ex typographia Aoademiae Scientiarum, 1736. URL: https://archive.org/details/mechanicasivemot02eule/page/28/mode/2up. [10]

[4] Jeremy Gray. Goursat, Pringsheim, Walsh, and the Cauchy integral theorem. *Math. Intelligencer*, 22(4):60–66, 77, 2000. doi:10.1007/BF03026773. [23]

[5] John K. Hunter. An introduction to real analysis, 2014. URL: https://www.math.ucdavis.edu/~hunter/intro_analysis_pdf/intro_analysis.pdf. [17]

[6] D. E. Knuth. *Axioms and hulls*, volume 606 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1992. doi:10.1007/3-540-55611-7. [21]

[7] N. J. Lennes. Theorems on the Simple Finite Polygon and Polyhedron. *Amer. J. Math.*, 33(1-4):37–62, 1911. doi:10.2307/2369986. [65]

[8] Jesper Lützen. *Joseph Liouville 1809–1882: master of pure and applied mathematics*, volume 15 of *Studies in the History of Mathematics and Physical Sciences*. Springer-Verlag, New York, 1990. doi:10.1007/978-1-4612-0989-8. [No citations.]

[9] John Sturgeon Mackay. *The Elements of Euclid: Books I to VI*. W. & R. Chambers, London and Edinburgh, 1884. [45]

[10] Reinhold Remmert. *Theory of complex functions*, volume 122 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, german edition, 1991. Readings in Mathematics. doi:10.1007/978-1-4612-0939-3. [27, 50]

[11] Reinhold Remmert. *Classical topics in complex function theory*, volume 172 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. Translated from the German by Leslie Kay. doi:10.1007/978-1-4757-2956-6. [No citations.]

[12] Rubí E. Rodríguez, Irwin Kra, and Jane P. Gilman. *Complex analysis*, volume 245 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013. In the spirit of Lipman Bers. doi:10.1007/978-1-4419-7323-8. [17, 50]

[13] Donald Sarason. *Complex function theory*. American Mathematical Society, Providence, RI, second edition, 2007. doi:10.1090/mbk/049. [1, 27]

[14] Elias M. Stein and Rami Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003. [6, 38, 41]

[15] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.4)*, 2025. https://www.sagemath.org. [No citations.]